Optimal selection of a common subset of covariates for different regressions

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Abstract

Given a regression dataset of size \( n \), most of the classical model selection literature treats the problem of selecting a subset of covariates to be used for prediction of future responses. In this paper we assume that a sample of \( J \) regression datasets of different sizes from a population of datasets having the same set of covariates and responses is observed. The goal is to select, for each \( n \), a single subset of covariates to be used for prediction of future responses for any dataset of size \( n \) from the population (which may or may not be in the sample of datasets). The regression coefficients used in the prediction are estimated using the \( n \) observations consisting of covariates and responses in the sample for which prediction of future responses is to be done, and thus they differ across different samples. For example, if the response is a diagnosis, and the covariates are medical background variables and measurements, the goal is to select a standard set of measurements for different clinics, say, where each clinic may estimate and use its own coefficients for prediction (depending on local conditions, prevalence, etc.). The selected subset naturally depends on the sample size \( n \), with a larger sample size allowing a more elaborate model. Since we consider prediction for any (or a randomly chosen) dataset in the population, it is natural to consider random covariates. If the population consists of datasets that are similar, our approach amounts to borrowing information, leading to a subset selection that is efficient for prediction. On the other hand, if the datasets are dissimilar, then our goal is to find a “compromise” subset of covariates for the different regressions. In addition, we propose a new measure, the generalized equivalent number of observations (GENO), which quantifies the usefulness of statistical models for prediction. Given a sample of size \( n \) and a subset of covariates denoted by \( p \), GENO compares the sample sizes needed for other subsets to achieve the same prediction error as \( p \), allowing the user to decide between different subsets of covariates.

Key words: prediction error, random covariates, Mallows \( C_p \), equivalent number of observations (ENO), GENO
1. Introduction

The classical theory of model selection in regression deals with the selection of a subset of covariates (aka features) that are useful for prediction. Numerous model selection methods have been suggested; AIC (Akaike [1]), Mallows $C_p$ (Mallows [12]), and FIC (ClAESKens and Hjort [5]) are prominent examples. These methods apply to a single regression dataset of a given size, for which a model is to be selected and then used for prediction. For a well-known Bayesian approach to model selection, see Schwarz [19]. A large body of literature emerged following these articles. In the setup of a single dataset, serious issues of optimality arise; see, e.g., Yang [24].

We consider the terms model selection and variable selection as identical, and we may use model for covariate subset. We study a model selection problem with data consisting of $J$ regression datasets (to which we may refer also as regression samples or just datasets or samples) that share the same random covariates and response variables. The datasets are given by $D_j := \{(X_{ij}, Y_{ij})\}, i = 1, \ldots, N_j, \ j = 1, \ldots, J$, where $X_{ij} \in \mathbb{R}^d$ is a column vector of $d$ random covariate values of the $i$th subject in the $j$th dataset, and $Y_{ij} \in \mathbb{R}$ is a response variable. For each $j$, the $N_j$ vectors $(X_{ij}, Y_{ij})$ are iid from some distribution $G_j \in \mathcal{G}$, where $\mathcal{G}$ is a set (population) of distributions. On the basis of these data we select a subset of covariates $p = p(n)$ to be used for the prediction of future responses for any randomly selected dataset of size $n$. Specifically, consider a regression dataset of size $n$, $D_J = (X_{iJ}, Y_{iJ}), \ i = 1, \ldots, n$, distributed according to $G_J$, which is randomly chosen from $\mathcal{G}$. The goal is to predict a new response $Y$ from the corresponding covariate vector $X$, where $(X, Y) \sim G_J$, using the selected set of covariates $p = p(n)$. The coefficients of these covariates to be used in the prediction are estimated using the dataset $D_J$. Since the dataset $D_J$ is random, it is natural to consider random covariates.

We assume that the set of distributions $\mathcal{G}$, which may be finite, or a hierarchical or superpopulation model, is sufficiently homogeneous in a way that justifies a common model selection. The data $D_J$ may be either one of the datasets $D_j$’s where $n = N_j$, or a new dataset. Clearly, the selected subset depends on $n$, with larger $n$ allowing larger subsets. Pooling together our $J$ regressions we select a single model; however, we allow different coefficients for different regressions. Our treatment of random covariates is based on a generalization of Mallows $C_p$ to random covariates that was inspired by notes generously given to us by Larry Brown (see [2]).

The motivating example we study concerns predicting the duration of visits of patients to specific doctors, given certain covariates that depend on the patients and the nature of the visit. We want to select a model such that for a doctor with $n$ observed visits ($n$ may be $N_j$ if this doctor is in our given data), we can efficiently predict the duration of a future visit.

There are two reasons for choosing a common model for different regressions. First, model selection requires large samples. Each sample size $N_j$ may be too small for model selection for
each regression separately, and we want to borrow information by pooling the \( J \) datasets. This is relevant when the different datasets arise from models that are similar. Second, in a variety of situations there is a need to choose a standard common set of covariates to be used for prediction. In this case we are trying to select a compromise model that can be used for the different regressions (and may not be optimal for some or any of them). For example, a large health organization may want to recommend a common standard set of tests for the purpose of certain diagnoses. The regression coefficients used for prediction based on this common set of tests may differ between countries or communities. Concerning economics models, the OECD attempts to standardize sets of common economic predictors to be used by its member countries. It makes sense to assume that in different countries, these predictors may have different weights.

We shall consider two situations. (a) The \( J \) datasets represent a finite population of size \( J \). For example, the OECD countries (in 2020) are a finite population of size \( J = 37 \), and inference refers only to these countries. (b) The \( J \) regression datasets form a sample from a larger or infinite population defined by a suitable probability space of distributions \( \mathcal{G} \), to be discussed in detail later. We shall consider the cases of \(|\mathcal{G}| = 1\), \(|\mathcal{G}| < \infty\), and \(|\mathcal{G}| = \infty\). We wish to choose a model that will be useful for prediction for both in-sample and also out-of-sample distributions \( G_j \).

Given the data \( D_j \) from distributions \( G_j \), \( j = 1, \ldots, J \), let \( m_j(X_{ij}) := E(Y_{ij} | X_{ij}, G_j) \) be the conditional expectation under \( G_j \). We do not assume a linear model but we shall approximate \( m_j(X_{ij}) \) by a linear function \( X_{ij}' \beta_j \), where \( \beta_j \) is the vector of projection coefficients under \( G_j \). We allow different \( \beta_j \)’s; however, our goal is to choose for each \( n \) a single subset of covariates. We shall require minimal assumptions on \( G_j \) as in White [23]. Our setup requires the covariates to be considered as random since we want to be able to choose models for prediction with covariates that will be observed in the future. When \( m_j(X_{ij}) \) is not linear then \( X_{ij} \) is not ancillary, and its marginal distribution matters; see, e.g., Buja et al. [3]. In this case conditioning on \( X \) or considering it as nonrandom leads to loss of information. For a recent discussion on fixed versus random \( X \) in the context of model selection see Rosset and Tibshirani [16]. When \( m_j(X_{ij}) = X_{ij}' \beta_j \), that is, when linearity is assumed, this model is called heterogeneous regression ANCOVA; see, e.g., Rutherford [17], Chapter 8, and the references therein. Related models appear under titles such as repeated measure regression (see, e.g., [22]), often with mixed effects. Our setup consists of \( J \) separate regressions with random covariates and fixed coefficients, and the new direction is the selection of a common model, that is, a set of covariates to be used for prediction.

Given a sample \( \{X_{ij}, Y_{ij}\} \) from \( G_j \), consider the model \( p \), that is, a subset of the covariates, denoted by \( p \). Let \( X_{ij}^{(p)} \) denote the subvector of \( X_{ij} \) consisting of the covariates in \( p \). Let \( \beta_j^{(p)} \) denote the linear projection coefficient vector and let \( \hat{\beta}_j^{(p)} \) be its least squares estimator, based on \( n \) observations. We assume that we are given \( n \) pairs of covariates and responses from a random distribution \( G_J \) sampled from \( \mathcal{G} \). For random variables \((X, Y)\) independent of the above samples
and distributed according to $G_J$, the linear prediction error is

$$R(n,p) := EE[(Y - (X^{(p)}/\hat{\beta}_{J,n}^{(p)})^2)|G_J],$$

where the inner expectation is over the randomness of $\hat{\beta}_{J,n}^{(p)}$ and $(X,Y)$ given $J = j$, and the outer expectation is over the randomness of $G_J$ in $\mathcal{G}$. The quantity $R(n,p)$ represents the prediction error associated with a given linear model $p$ and a random dataset of size $n$. Our goal is to estimate $R(n,p)$ and related quantities, in order to select models for prediction on the basis of $n$ observations.

To quantify the relative quality of a model, we introduce a new measure, GENO, which generalizes the measure ENO (equivalent number of observations) appearing in Erev, Roth, Slonim, and Barron [7]. GENO can be described in terms of $R(n,p)$ as follows: given a model $p$ with $n$ observations and another model $q$, define GENO($n;p,q$) to be the value of $m$ satisfying $R(m,q) = R(n,p)$. In words, GENO($n;p,q$) is the number of observations required in order for model $q$ to predict equally well as model $p$, when the parameters of the latter model are estimated on the basis of $n$ observations. In Section 4 we shall use an approximation to $R(n,p)$ to define and estimate GENO. Such a measure allows us to decide between a set of covariates that may be good for prediction but costly to obtain, and another set of more accessible covariates that we may consider using even if their predictive value is lower and therefore it may require more observations. A comparison in terms of the sample size required by one model (for prediction, testing or estimation) to be as good as another with a given sample size is closely akin to the notion of Pitman efficiency; see, e.g., Zacks [25]. Our approach to quantifying the value of a model is close in spirit, but not in detail, to the work of Lindsay and Liu [11] who define a “model credibility index” as the sample size $N^*$, where data from the model and from the true generating process are indistinguishable in the sense that for a given goodness of fit test of the model with $N^*$ observations, the probability of rejection under the model is, say, 50%.

In Section 2 we define the problem and the basic results and notation for a single regression dataset, as a preliminary to the main part, Section 3 where we consider model selection for several regression datasets. In Section 4 we discuss the GENO measure of the relative quality of models. In Section 5 we demonstrate the results by simulations, and in Section 6 we discuss an application to a medical management problem of predicting service times, that is, visit durations of patients in hospital. Section 7 is an appendix that contains all the proofs. Appendix B summarizes the notation used in the paper.

2. Prediction error with random covariates: A single sample

We start with the case $|\mathcal{G}| = J = 1$, that is, with selection of a model for prediction given a single regression dataset. This is the case treated in the standard model selection literature. In this
case the distinction between \( N = N_1 \) and \( n \) may seem artificial. However, we consider this case as a starting point. Therefore, for now we assume that we observe a sample of size \( N \): \( \{(X_i, Y_i)\} \) of iid pairs from some distribution \( G \) that we use as a training set for model selection, that is, for choosing a model for prediction of a future \( Y \) from \( X \), with parameters that will be estimated using \( n \) observations from \( G \).

We derive some results that will be needed for the general case \( J > 1 \), to be discussed in Section 3. Subsets of the covariates are denoted by \( p, q \), etc. With a slight abuse of notation we denote the size of the set also by \( p \), rather than \( |p| \). We refer to the associated linear model as model \( p \). For now, we fix \( p \) and suppress it in our notation. Later we shall distinguish between the set of \( d \) of all covariates, that is, \( X \in \mathbb{R}^d \), and subsets of covariates denote by \( p \) or \( q \).

2.1. Preliminaries

Consider a dataset \( D = \{(X_i, Y_i)\} \) of iid pairs from some distribution \( G \), where \( X_i \) is a column vector in \( \mathbb{R}^p \), \( i = 1, \ldots, n \). Let \((X, Y)\) without indexes denote one such “generic” observation, distributed independently of the dataset \( D \) as any \((X_i, Y_i)\). The first entry of each \( X_i \) may be 1, so that the models may include an intercept term.

We set \( Q := E(XX') \). When we want to emphasize the subset \( p \) of variables we write \( Q^{(p)} := E(X^{(p)}X^{(p)'}). \) Let \( Y_n \in \mathbb{R}^n \) denote the \( n \)-column vector of the \( Y_i \)'s. We do not assume a linear model, and set \( E(Y|X) := m(X) \) for some function \( m \). Assuming that both \( X \) and \( Y \) have finite second moments and that \( Q \) is invertible, the best linear approximation of \( m(X) \) is \( X'\beta \), where

\[
\beta := \arg \min_{b \in \mathbb{R}^p} E\{m(X) - X'b\}^2 = Q^{-1}E(XY),
\]

and the same projection coefficient vector \( \beta \) also satisfies \( \beta = \arg \min_b E(Y - X'b)^2 \); hence \( X'\beta \) is the best linear predictor of \( Y \). Our assumptions imply that the minimizer \( \beta \) is unique. Set

\[
e_i := Y_i - X'_i\beta, \quad \text{where } \beta \text{ is defined in } (2).
\]

By (2.25) in Hansen [9], we have \( E(Xe) = 0 \), where again \( X \) and \( e \) are “generic” \( X_i \) and \( e_i \).

Define \( X_n \) to be the \( n \times p \) matrix whose \( n \) rows are the rowvectors \( X_i' \). In this common notation, the standard linear model will be written as \( X_n\beta \), whereas each of its rows as \( X_i'\beta \), and \( X_n'X_n = \sum_{i=1}^n X_iX_i' \). Most of our notation and the standard results we use can be found, for example, in Hansen [9]. Under standard assumptions, the least squares estimator, that is, the value of \( b \) that minimizes \( ||Y_n - X_n b||^2 \), is

\[
\hat{\beta}_n = (X_n'X_n)^{-1}X_n'Y_n.
\]

The assumption that \( (X_n'X_n)^{-1} \) exists (with probability 1) holds if we assume that \( X \) has a continuous distribution. For the existence of certain moments required later we shall assume that the distribution of \( X \) is a mixture of normals. See Hansen [9], pp. 102–3, for a discussion of the
existence of $(X_n'X_n)^{-1}$ and its moments. Without assuming continuity, the assumption that $Q$ is invertible implies that $(X_n'X_n)^{-1}$ exists with probability converging to 1 as $n \to \infty$; however, for discrete distributions this probability is less than one, and thus $\hat{\beta}_n$ may not exist, and has no finite moments, a “conundrum” in the words of Hansen. In Section 2.3 we extend our discussion to discrete covariates by conditioning on the existence of a bounded inverse, and showing that under simple conditions this amounts neglecting a set having an exponentially small probability. This provides some solution to the above conundrum.

We now assume that $(X_n'X_n)^{-1}$ exists and has sufficiently many moments so that expressions like (5) below are finite. If $X$ and $Y$ have finite fourth moments, then by Theorem 7.3 in Hansen

\[ \sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0, Q^{-1}WQ^{-1}), \]  

where $W := E(XX'e^2)$, a $p \times p$ matrix assumed to be positive definite. Starting with a single distribution $G$ and a sample $D$ as above, the prediction error of (1) incurred by model $p$ based on a sample of size $n$ is

\[ R(n,p) := E(Y - X'\hat{\beta}_n)^2, \]  

with $X = X^{(p)}$, that is, the vector consisting of the covariates in $p$, and the response $Y$ being independent of $\hat{\beta}_n = \hat{\beta}_n^{(p)}$. When we consider several models, we invoke the model $p$ and set, for example, $W^{(p)} := E(X^{(p)}X^{(p)'e^2})$, and likewise for $Q$, etc.

### 2.2. Equally good sequences of models

When selecting the best model for a given $n$, that is, the subset of covariates that minimizes $R(n,p)$, we should take into account that different samples yield different estimators $\hat{\beta}_n$, leading to different prediction errors; thus, there is no gain in optimizing more precisely than the difference between such errors. Consider the prediction error conditioned on the estimated regression coefficients

\[ R\left(n,p;\hat{\beta}_n\right) := E \left[ \left( Y - X'\hat{\beta}_n \right)^2 \right| \hat{\beta}_n. \]  

Note that $R(n,p) = E\{R(n,p;\hat{\beta}_n)\}$. Writing $\left( Y - X'\hat{\beta}_n \right)^2 = \left( Y - X'\beta + X'\beta - X'\hat{\beta}_n \right)^2$, a straightforward expansion, taking conditional expectation, and using $E[(Y-X'\beta)X'\hat{\beta}_n] = E[eX'] = 0$ shows that

\[ R\left(n,p;\hat{\beta}_n\right) = E(Y - X'\beta)^2 + E \left[ \left( X'(\hat{\beta}_n - \beta) \right)^2 \right| \hat{\beta}_n. \]  

The first term in (6) is a constant and the second term equals $(\hat{\beta}_n - \beta)'Q(\hat{\beta}_n - \beta)$, which is of order $O_p(1/n)$ since $\sqrt{n}(\hat{\beta}_n - \beta) = O_p(1)$; see (4). This means that when we condition on the observed $\hat{\beta}_n$, the prediction errors of two different samples differ by a term of order $O_p(1/n)$. Hence, if two sequences of models $p(n)$ and $q(n)$ satisfy

\[ |R(n,p(n)) - R(n,q(n))| = o(1/n) \text{ or equivalently } \lim_{n \to \infty} n|R(n,p(n)) - R(n,q(n))| = 0, \]
we consider them to be \textit{equally good}. If \( p = p(n) \) is best in the sense of minimizing \( R(n, p(n)) \) and \( q = q(n) \) is equally good, we say that \( q \) is \textit{adequate}, and rather than choose “best models” we settle for a adequate models. See, e.g., Nevo and Ritov \cite{13} for a related approach.

2.3. Versions of Mallows \( C_p \) for random covariates

Given a dataset \( D = \{ (X_i, Y_i) \} \) of size \( N \), we first estimate the prediction error \( E \) incurred if prediction is to be based on \( n \) observations. We shall consider two types of asymptotics: one when \( n \) is considered to be large, and the other when \( n \) is fixed, and \( N \) is large. Asymptotics in \( J \) will be considered later.

We use the following notation: set \( \hat{Q}_N := \frac{1}{N}X_NX_N' \), and let \( Y_N \) denote the \( N \)-vector of the \( Y_i \)'s. Recalling the notation \( e_i = Y_i - X_i' \beta \), let \( e_N \) denote the \( N \)-vector having components \( e_i \). Set \( \hat{W}_N := \frac{1}{N} \sum_{i=1}^{N}X_i'X_i' \) with \( \hat{e}_i = Y_i - X_i' \hat{\beta}_N \), where \( \hat{\beta}_N \) is given by (\ref{3}) upon replacing \( n \) by \( N \). Thus in \( (8) \) below, \( \frac{1}{n}||Y_N - X_N' \hat{\beta}_N||^2 = \frac{1}{N} \sum_{i=1}^{N} e_i'^2 \). Let \( V := W \hat{Q}^{-1} \), and \( \hat{V}_N := \frac{1}{N}X_N'X_N'e_N = \frac{1}{N} \sum_{i=1}^{N}X_i'e_i \). Note that \( U_N := \frac{1}{N}X'N'e_N = \frac{1}{N} \sum_{i=1}^{N}X_i'e_i \). We define the approximate prediction error to be

\[
AR(n,p) := E(Y - X' \beta)^2 + \frac{1}{n} tr(V),
\]

where \( tr \) denotes trace, and equation (\ref{9}) of Theorem \( 2.1 \) below shows that it is an approximation to the quantity \( R(n,p) \) of (\ref{5}). Clearly \( E(Y - X' \beta_N)^2 > E(Y - X' \beta)^2 \) and the trace is an approximation of the difference up to a term of order \( O(1/n^{3/2}) \), as explained in (\ref{32}) and the ensuing calculations. Next we define the statistic \( C^{(p)}(n,N) \) as an estimator of \( AR(n,p) \) by

\[
C^{(p)}(n,N) := \frac{1}{N}||Y_N - X_N' \hat{\beta}_N||^2 + \frac{1}{n} tr(\hat{V}_N) \left( \frac{1}{n} + \frac{1}{N} \right).
\]

The new term \( \frac{1}{n} tr(\hat{V}_N) \) is an approximately (up to \( o_p(1/N) \)) unbiased estimator of \( \frac{1}{N}||Y_N - X_N' \hat{\beta}_N||^2 - E(Y - X' \beta)^2 \), as shown in (\ref{35}) and (\ref{36}). The fact that \( tr(\hat{V}_N) \) is a biased estimator of \( tr(V) \) entails a bias of order \( 1/n \) for the estimator \( C^{(p)}(n,N) \) as an estimator of \( AR(n,p) \). We shall study the latter estimator, and when we use it, we shall apply a standard jackknife correction for its bias; see Efron \cite{6}, Equation (2.8). We denote the bias-corrected \( C^{(p)}(n,N) \) by \( C^{(p)}(n,N) \).

The superscript \( p \) in \( C^{(p)} \) is a reminder of the suppressed model notation, and in particular \( X_i \in \mathbb{R}^p \). This notation will become relevant when we use subvectors of covariates of different dimensions. The statistic \( C^{(p)}(n,N) \) is a counterpart of Mallows \( C_p \), but here we consider random covariates. Furthermore, we distinguish between the number \( N \) of observations used for the choice of the model and the sample size \( n \) of observations used for estimating the model’s parameters. The classic Mallows \( C_p \) concerns nonrandom covariates, where \( n = N \), and the true model is assumed to be linear.
To see the relation to Mallows $C_p$, assuming a linear model for the subset of covariates $p$, we have that $e_i = Y_i - X_i'\beta$ is independent of the covariates, with variance $\sigma^2$, and $W = \sigma^2 Q$, and therefore $\hat{\mathcal{V}}_N = \hat{\mathcal{W}}_N(\hat{Q}_N)^{-1}$ will converge to $\sigma^2 I_p$ and $\text{tr}(\hat{\mathcal{V}}_N)$ to $\sigma^2 p$. If we use $\sigma^2 p$ as an approximation of $\text{tr}(\hat{\mathcal{V}}_N)$ (and therefore we only have to estimate $\sigma^2$ rather than a trace), then $C(p)$ in the case $N = n$ coincides with Mallows $C_p$.

The following theorem provides the rate of approximation of $AR(n, p)$ to $R(n, p)$, and then analyzes $C(p)$ as an estimator of $AR(n, p)$; some of its conditions and implications are discussed below. All proofs are in the Appendix. Our proof shows that Assumption (i) below can be replaced by the assumption that $X$ and $Y$ have 24 finite moments, and a careful inspection of the proof shows that this number can be somewhat reduced.

**Theorem 2.1.** Assume that

(i) The coordinates of $X$ and $Y$ have finite moments of all orders.

(ii) The entries of $(X_n'X_n/n)^{-1}$ have third moments that are bounded uniformly in $n$. Then

$$|R(n, p) - AR(n, p)| = O(1/n^{3/2}),$$

and

$$AR(n, p) - C(p)(n, N) = \mathcal{E}_N + \frac{1}{n} \left\{ \text{tr}(\mathcal{V}) - \text{tr}(\hat{\mathcal{V}}_N) \right\} + o_p(1/N),$$

where

$$\mathcal{E}_N = E(Y - X'\beta)^2 - \frac{1}{N}||Y_N - X_N\beta||^2 + \frac{1}{N} \left\{ \text{tr}(U_N^UQ^{-1}_N) - \text{tr}(\mathcal{V}) \right\}.$$  

Furthermore,

(i) $\mathcal{E}_N = O_p(1/\sqrt{N})$,

(ii) $\text{tr}(\mathcal{V}) - \text{tr}(\hat{\mathcal{V}}_N) = O_p(1/\sqrt{N})$

and

$$\sqrt{N}(C(p)(n, N) - AR(n, p)) \xrightarrow{D} N(0, \sigma^2)$$

for some asymptotic variance $\sigma^2$ as $N \to \infty$, and $n$ is fixed.

Since there is only a finite number of models, the above constants in $O$, $O_p$, and $o_p$ do not depend on the model $p$. For example, we could replace [9] by $|R(n, p) - AR(n, p)| \leq B/n^{3/2}$ for all $n$ and $p$, where $B$ is a constant. Moreover, the term $o_p(1/N)$ in [10] does not depend on $n$.

Condition (i) of Theorem of 2.1 is standard, and Lemma 2.2 below shows that Condition (ii) is satisfied if $X$ is distributed as a mixture of normals; see Sampson [18]. Such mixtures form a dense family of distributions with respect to weak convergence in the space of distribution on $\mathbb{R}^p$. As the distribution of $X$ is never known exactly, it makes sense to assume, as an approximation, that the data satisfy such a condition. The case where $X$ has discrete components is discussed in Section 2.4.
We shall compare models consisting of different subsets of covariates \( p \). Equation (9) suggests that choosing a model \( p \) by minimizing a good estimate of \( AR(n, p) \) with respect to \( p \) can lead to a model for which \( R(n, p) \) is within \( o(1/n) \) of the best model, and thus \( p \) is an adequate model in the sense of Section 2.2. This is stated formally in Proposition 2.4.

In view of (10) we use \( C(p)(n, N) \) of (8) as an estimator of the approximate prediction error \( AR(n, p) \) and hence of the prediction error \( R(n, p) \). This is formalized in Propositions 2.5 and 2.6 below. We now briefly discuss Equations (10) and (11). First consider the bias of \( C(p)(n, N) \) as an estimator of \( AR(n, p) \). It is easy to see that \( E\mathbb{E}_N = 0 \). By (12) (ii), \( tr(V) - tr(\hat{V}_N) = O_p(1/\sqrt{N}) \), and after dividing the latter term by \( n \) as in (10), it is of a smaller order than the term \( tr(\hat{V}_N) \left( \frac{1}{n} + \frac{1}{N} \right) \) appearing in \( C(p)(n, N) \). This shows that the latter term contributes to reducing the bias of \( C(p)(n, N) \) as an estimator of \( AR(n, p) \). Our main interest is in the case of \( J \) regressions, and in choosing a model that minimizes an average of \( J \) \( AR \) values. Averaging (nearly) unbiased estimates can result in consistency in \( J \), which explains why we care about correcting the bias of \( C(p)(n, N) \). In this case, a further bias correction using the jackknife is useful (see Section 5.2).

Choosing a good model can be reduced to choosing between two models, say, \( p \) and \( q \) at a time, by approximating the difference \( AR(p) - AR(q) \) using \( C(p)(n, N) - C(q)(n, N) \). The leading terms in the latter expression will be the difference between the relevant values of \( \mathcal{E}_N \) for the two models, and it is easy to see that the leading term of this difference is the difference between the values of \( \frac{1}{N} ||Y_N - X_N\beta||^2 \) for the corresponding models, which is of order \( O_p(1/\sqrt{N}) \) by the central limit theorem. However, when two models having very similar prediction values are compared, their leading terms will approximately cancel, and in this case the second term on the right-hand side of (8) plays a role. This holds also for Mallows \( C_p \) and the AIC, [1], and will be exploited formally in the propositions below.

The following lemma shows that Condition (ii) of Theorem 2.1 holds when \( X \) is distributed as a mixture of normals.

**Lemma 2.2.** Let the distribution of the covariate vector excluding the first coordinate, 1, be normal, or a finite mixture of normals, or an infinite mixture of normals with covariance matrices in a set \( \Xi \), and \( \inf_{\Sigma \in \Xi} \lambda_{\min}(\Sigma) > 0 \), where \( \lambda_{\min} \) denotes the smallest eigenvalue. Then, for \( n > p + 5 \), Condition (ii) of Theorem 2.1 is satisfied.

We remark that more generally, the \( r \)th moments of the entries of \( (X_n'X_n/n)^{-1} \) are bounded under the conditions of Lemma 2.2 provided that \( n > p + 2r - 1 \) (see von Rosen [15], Theorem 4.1). Note that the condition on \( \lambda_{\min} \) guarantees that \( X \) is bounded away from multicollinearity.

### 2.4. Discrete covariates

When \( X \) contains discrete covariates, the matrix \( (X_n'X_n/n)^{-1} \) does not exist with positive probability, and expressions like \( \hat{\beta}_n \) of (8) and hence \( R(n, p) \) of (5) do not exist. When the
components of $X$ are bounded, we provide the following limiting approach. Set
\[
B_n := \left\{ X_n : \lambda_{\min}(X'_n X_n/n) < \frac{1}{2} \lambda_{\min}(Q) \right\}, \quad \text{and} \quad G_n := B_n^c. \tag{14}
\]
Define $\bar{R}(n,p) = E[(Y - X' \hat{\beta}_n)^2 | G_n]$. We have

**Theorem 2.3.** Suppose that $Y$ has all moments, the components of $X$ are bounded, and $Q$ is invertible; then for some $a < 1$,
\[
|\bar{R}(n,p) - AR(n,p)| = O(1/n^{3/2}) \quad \text{and} \quad P(G_n) > 1 - a^n \lambda_{\min}(Q).
\]
Moreover, all quantities appearing in Theorem 2.1 are well defined on $G_N$, and can be defined in an arbitrary way outside of $G_N$, and the results (10)–(13) hold.

Thus, apart from the set $B_n$, which has exponentially small probability, the approximation rate of $AR(n,p)$ to the prediction error is the same as in (9) and the rest of Theorem 2.1 still holds. The result follows from Theorem 2.1 and Lemma 2.3 given in the Appendix.

2.5. Approximations and consistency

The focus of this paper is on choosing a good model for prediction of future responses with parameters that are estimated from a sample of size $n$, with the understanding that different sample sizes may (and should) lead to different choices of models; more specifically, a larger $n$ allows for a larger set of covariates. The choice of a model may be based on the same $n$ observations, or on another sample of size denoted by $N$. Since the chosen model depends on $n$, asymptotic results in $n$ are not of major interest in this context; however, they may contribute some understanding when $n$ is not small. Such results are discussed in this section.

In Proposition 2.5 we show that under the conditions of Theorem 2.1, choosing a model in the set $\arg\min_p C^{(p)}(n,N)$ guarantees that for increasing $n$ and $N$ we choose the best linear model with probability converging to 1, that is, the model minimizing $R(n,p) = E \left( Y - X^{(p)'} \hat{\beta}_n^{(p)} \right)^2$, with notation defined after (5). In Proposition 2.6 we show that for fixed $n$, using $C^{(p)}(n,N)$, we choose an adequate model in the sense defined in Section 2.2 with probability converging to 1 as $N \to \infty$.

For a given $n$, define the following sets:
\[
p^*(n) := \arg\min_p R(n,p) = \arg\min_p E \left( Y - X^{(p)'} \hat{\beta}_n^{(p)} \right)^2,
\]
\[
\pi^*(n) := \arg\min_p AR(n,p) = \arg\min_p \left\{ E \left( Y - X^{(p)'} \hat{\beta}_n^{(p)} \right)^2 + \frac{1}{n} \text{tr}(V_n) \right\},
\]
\[
p^* := \arg\min_{p \in \mathcal{M}} |p|, \quad \text{where} \quad \mathcal{M} := \arg\min_p E \left( Y - X^{(p)'} \beta^{(p)} \right)^2 \quad \text{and} \quad |p| \text{denotes the number of covariates in the model } p,
\]
\[
\hat{\pi}^*(n,N) := \arg\min_p C^{(p)}(n,N).
\]
The following proposition shows that the first two sets defined above by arg min converge to the third, which is a singleton. Note that \( p^* \) is the best linear model in the sense of being the most parsimonious model minimizing the expected square of the projection error \( Y - X^{(p')\beta(p')} \). We deal with the convergence of \( \hat{\pi}^*(n,N) \) in Proposition 2.5.

**Lemma 2.4.** Suppose that the conditions of Theorem 2.1 hold. Then

(i) Any two sequences in \( \pi^*(n) \) and \( p^*(n) \) are equally good, that is, any sequence of models in \( \pi^*(n) \) is adequate in the sense of Section 2.2.

(ii) The set \( p^* \) is a singleton, and the sets \( \pi^*(n) \) and \( p^*(n) \) converge to the singleton \( p^* \) as \( n \to \infty \).

The proof shows that essentially \( M \) is a singleton; that is, besides \( p^* \), \( M \) may only contain models having the same regression coefficients as those of \( p^* \), and further coefficients that vanish.

Note that since the number of models is finite, it follows that \( \hat{\pi}^*(n,N) = \pi^*(n) = p^*(n) \) for large enough \( n \) and \( N \); that is, the same model \( p^* \) minimizes both \( R(n,p) \) and \( AR(n,p) \). The model \( p^* \) is the minimal best linear predictive model that one would ideally use if the projection coefficients \( \beta^{(p)} \) were known.

The next proposition provides asymptotics that lead to the correct selection of the best model, that is, the model that minimizes the prediction error \( R(n,p) \), with probability converging to 1.

**Proposition 2.5.** Under the conditions of Theorem 2.1 with both \( n,N \to \infty \), and \( n/N \to 0 \), we have \( P(\hat{\pi}^*(n,N) = p^*(n)) \to 1 \).

The proof is given in the Appendix, where we also show that the condition \( n/N \to 0 \) is necessary. The case \( n = N \) (with nonrandom covariates) corresponds to the standard Mallows \( C_p \), which is inconsistent; more specifically, it is well known that for \( n = N \), the choice \( \hat{\pi}^*(n,N) \) may lead to models \( q \) that strictly contain \( p^* \); see, e.g., Nishii [14]. The equality \( \hat{\pi}^*(n,N) = p^*(n) \), which holds for large enough \( n \) and \( N \) with high probability, implies that \( \hat{\pi}^*(n,N) \) is a singleton (by Lemma 2.4 (ii)), and that selecting a model according to the statistic \( \hat{\pi}^*(n,N) \) yields a model that minimizes the prediction error. Furthermore, the choice of a model by \( \hat{\pi}^*(n,N) \) leads asymptotically to the choice of \( p^* \), the smallest model in terms of the number of covariates in \( M \), that is, the most parsimonious model \( p \) that minimizes \( E(Y - X^{(p')\beta(p')})^2 \). Convergence of the selected model to that model is often referred to as consistency; see, e.g., Shao [20].

In the case of fixed \( n \), Equation (13) readily implies that \( C^{(p)}(n,N) - AR(n,p) = O_p(1/\sqrt{N}) \). Therefore, as \( N \) goes to infinity, the left-hand side converges to zero (at a rate of \( 1/\sqrt{N} \)), implying

**Proposition 2.6.** Under the condition of Theorem 2.1, we have for any fixed \( n \),

\[
P \left( \hat{\pi}^*(n,N) \subseteq \pi^*(n) \right) \to 1.
\]

In words, Proposition 2.6 says that a model that minimizes \( C^{(p)} \) will minimize \( AR(n,p) \) with high probability for fixed \( n \) and a suitably large \( N \). Lemma 2.4 (i) asserts that minimizing \( AR(n,p) \)
(by \(\pi^*(n)\)) is close to minimizing \(R(n,p)\) (by \(p^*(n)\)), which is our goal.

3. Several samples

Our main focus is on the case where several regression datasets are observed. We first discuss the case where we observe samples from all the regressions of interest, and then, in Section 3.3, we consider a hierarchical situation where the data consist of a random sample of regression datasets from a structured collection of regression models.

3.1. Model selection observing all regressions

We consider a population of distributions \(\mathcal{G} = \{G_j : j = 1, \ldots, J\}\) of size \(J\), and observe data \(D_j = \{(X_{ij}, Y_{ij}) \sim G_j, i = 1, \ldots, N_j\}, j = 1, \ldots, J\), and \(X_{ij} \in \mathbb{R}^d\). We assume that these \(J\) distributions comprise the whole population.

For a given \(n\), the goal is to select a common set of covariates \(p\) to be used for prediction of the response \(Y\) from \(X = X^{(p)}\) (the subvector with coordinates in \(p\)) for each individual distribution \(G_j\) from the population, where the coefficients \(\hat{\beta}_{j,n}\), which are allowed to vary with \(j\), are estimated with a sample of size \(n\). The relevant prediction error for this task is (16) below. When predicting for individual \(j\), it may be natural to set \(n = N_j\). However, other values of \(n\) may be of interest in studying the contribution of covariates as a function of the sample size. Later (in Section 3.3), we use the \(J\) samples as a training set for choosing a model to predict for any out-of-sample \(G_J\) on the basis of \(n\) future observations, where \(n\) is not determined in advance since \(J\) is not in the training set. In this case we use the chosen model, and estimate its parameters on the basis of a dataset of size \(n\) from \(G_J\). The value of \(n\) may vary, being the size of the dataset that can be obtained from \(G_J\).

In our motivating example (see Section 6) we have data consisting of a sample of \(J\) doctors (more generally, servers) from a given hospital, and for the doctor indexed by \(j\), we have a sample of durations of visits (service times) \(Y_{ij}\) of \(N_j\) patients (customers) along with a set of covariates \(X_{ij}\) having the joint distribution \(G_j\). The goal is to select one model, that is, a set of covariates, that will be used to predict a new patient’s visit duration for any doctor in the population, who may or may not be in our sample, rather than to consider a different set of covariates for each doctor.

Let \(X := X^{(p)} \in \mathbb{R}^p\), where for now \(p\) is suppressed. For each \(j\) and generic observation \((X,Y)\) from the distribution \(G_j\), we define

\[
\beta_j := \arg\min_{\beta} E_{G_j}(Y - X'\beta)^2 = Q_j^{-1}E_{G_j}(XY);
\]
see [2], where \( Q_j := E_{G_j}(XX') \). Assuming finite fourth moments, we have for a sample size \( n \to \infty \), for each \( j \), as in [4],
\[
\sqrt{n}(\hat{\beta}_{j,n} - \beta_j) \xrightarrow{D} N(0, Q_j^{-1}W_jQ_j^{-1}),
\]
where \( X_{j,N} \) and \( Y_{j,N} \) are the \( j \)th versions of \( X_N \) and \( Y_N \), and \( W_j := E_{G_j}(XX'e^2) \), a \( p \times p \) matrix, is assumed to be positive definite. We further use the notation \( V_j \) for the \( j \)th version of \( V \), that is, when expectations are taken with respect to \( G_j \), and similar notation when \( N = N_j \) observations are used for the estimators \( \hat{Q}_{j,N}, \hat{V}_{j,N}, \) and \( \hat{W}_{j,N} \) instead of \( Q_N, V_N, \) and \( W_N \).

We consider prediction for a random individual from the population \( G \), and therefore the relevant prediction error is
\[
R(n,p) := \frac{1}{J} \sum_{j=1}^J R_j(n,p) := \frac{1}{J} \sum_{j=1}^J E_{G_j}(Y - X'b_{j,n})^2,
\]
where \( (X,Y) \sim G_j \) independently of \( \hat{\beta}_{j,n} \), and the expectation on the right-hand side of \( \text{(16)} \) is also applied to \( \hat{\beta}_{j,n} \). In \( R(n,p) \) above and similar expressions below, we suppress the number of datasets \( J \). In the case that any of the distributions \( G_j \) involves discrete covariates, we replace \( E_{G_j}(Y - X'b_{j,n})^2 \) by a conditional expectation as in Section \( 2.4 \) where the conditioning is on a set whose complement is exponentially small. In the definition of Equation \( \text{(16)} \) and others below we use boldface letters when \( J > 1 \). Next define
\[
AR(n,p) := \frac{1}{J} \sum_{j=1}^J AR_j(n,p) := \frac{1}{J} \sum_{j=1}^J \left\{ E_{G_j}(Y - X'b_j)^2 + \frac{tr(V_j)}{n} \right\}.
\]
Using [9] we have \( R(n,p) = AR(n,p) + O(1/n^{3/2}) \). Set
\[
C_{j}(n,N_j) := \frac{1}{N_j} \left\| Y_{j,N_j} - X_{j,N_j}\hat{\beta}_{j,N_j} \right\|^2 + tr(\hat{V}_{j,N_j})(1/n + 1/N_j), \quad \text{and}
\]
\[
C(n,N) := \frac{1}{J} \sum_{j=1}^J C_{j}(n,N_j),
\]
where \( N = (N_1, \ldots, N_J) \). We define the jackknife bias-corrected \( C^{(p)} \) by
\[
C^{(p)}(n,N) := \frac{1}{J} \sum_{j=1}^J C_j^{(p)}(n,N_j),
\]
where \( C_j^{(p)}(n,N_j) \) is the bias-corrected \( C_j(n,N_j) \); see Efron [6], Equation (2.8).

Theorem 3.1 below parallels Theorem 2.1 concerning the error of \( C^{(p)}(n,N) \) as an estimator of \( AR(n,p) \).

**Theorem 3.1.** Suppose that the conditions of Theorem 2.1 are satisfied when \( (X,Y) \sim G_j \) for each \( j = 1, \ldots, J \). Then,
\[
AR(n,p) - C^{(p)}(n,N) = \frac{1}{J} \sum_{j=1}^J \mathcal{E}_{j,N_j} + \frac{1}{nJ} \sum_{j=1}^J \left\{ tr(V_j) - tr(\hat{V}_{j,N_j}) \right\} + \frac{1}{J} \sum_{j=1}^J \mathcal{O}_{p} \left( \frac{1}{N_j} \right),
\]
where $E_{j,N_j}$ is the $j$th version of $E_N$ defined in (11), and the $a_p$ terms do not depend on $n$.

Moreover, assume that $\lim_{N \to \infty} N_1/N_j := a_j$ exists for all $j$, where $0 < a_j < \infty$; then for any fixed $n$ and $J$,

$$\sqrt{N_1} \left\{ A_R(n,p) - C^{(p)}(n,N) \right\} \xrightarrow{D} N(0,\sigma^2_J),$$

as $N \to \infty$, where $\sigma^2_J = \frac{1}{J} \sum_{j=1}^J a_j \sigma^2_j$ and $\sigma^2_j$ is the asymptotic variance under $G_j$ as in Theorem 2.1, Equation (13).

Notice that if $\sigma^2_j$ and $a_j$ are of the order of a constant (in $j$), then the asymptotic variance of $\sqrt{N_1} \left\{ A_R(n,p) - C^{(p)}(n,N) \right\}$ decreases like $1/J$, which means that the error is decreasing in $J$.

### 3.2. Consistency

Similarly to Section 2.5, define

$$p^*(n) := \arg \min_{p} R(n,p) = \arg \min_{p} \sum_{j=1}^J E_{G_j}(Y - X^{(p)}\hat{\beta}_j^{(p)})^2,$$

$$\pi^*(n) := \arg \min_{p} AR(n,p) = \arg \min_{p} \sum_{j=1}^J \left\{ E_{G_j}(Y - X^{(p)}\hat{\beta}_j^{(p)})^2 + \frac{\text{tr}(V_j^{(p)})}{n} \right\}, \quad (20)$$

$$p^* := \arg \min_{p \in M} |p| \text{ where } M := \arg \min_{p \in M} \sum_{j=1}^J E_{G_j}(Y - X^{(p)}\beta_j^{(p)})^2,$$

$$\hat{p}^*(n,N) := \arg \min_{p} C^{(p)}(n,N).$$

Similar to Proposition 2.4 and essentially by the same proof, the following result holds.

**Proposition 3.2.** Suppose that the conditions of the first part of Theorem 3.1 hold. Then, the set $p^*$ is a singleton and the sets $p^*(n)$ and $\pi^*(n)$ converge to the singleton $p^*$ as $n \to \infty$ for any fixed $J$.

The following proposition generalizes Propositions 2.4 and 2.5 to several regression samples.

**Proposition 3.3.**

1. Assume that the conditions of the first part of Theorem 3.1 hold. Then for fixed $n$ we have

$$\lim_{N \to \infty} P \left( \hat{p}^*(n,N) \subseteq \pi^*(n) \right) = 1.$$

2. Let $n/N_j$ be bounded for all $j = 1, \ldots, J$, and let $C$ be a constant satisfying for all $j$ and $p$

$$n/N_j, \lambda_{\max}(W_j^{(p)}), 1/\lambda_{\min}(W_j^{(p)}), \lambda_{\max}(Q_j^{(p)}) \leq C. \quad (21)$$

Then

$$\lim_{n \to \infty} \inf_{N \to \infty} P \left( \hat{p}^*(n,N) = p^*(n) \right) \geq 1 - K_C/J, \text{ where } K_C \text{ depends only on } C.$$
The existence of $C$ follows from the assumption on $n/N_j$ since only a finite number of terms appear in (21) besides $n/N_j$.

Part 1 of the above proposition extends Propositions 2.6. Part 2 extends Proposition 2.5, however, a stronger condition was needed before, namely that $n/N \to 0$, to obtain consistency. Here, we obtain approximate consistency for large $\mathcal{J}$, assuming only that $n/N_j$ is bounded, along with the other terms in (21). This is useful since in our application it is natural to consider the possibility that $n = N_j$.

The rate $K/\mathcal{J}$ in the theorem was achieved by assuming appropriate second moments and using Chebyshev’s inequality. Assuming $2m$ moments leads in the same way to the rate $K/\mathcal{J}^m$, and with further assumptions, a large deviation bound can be achieved.

### 3.3. A population of distributions

We now consider the situation where we have a sample of $\mathcal{J}$ individuals from a given population, finite or infinite, and we are interested in predictions for a random (possibly out-of-sample) individual from the same population. In terms of the application considered in this paper, this situation corresponds to the case that we have a sample of $\mathcal{J}$ doctors out of many more, and our goal is to select a model to be used to predict service durations for a random doctor.

Formally, let $(\Theta, \mathcal{T}, \mathcal{P})$ be a probability space and let $\{G_\theta : \theta \in \Theta\}$ be a family of distributions. Thus, $G_\theta$ is a random probability measure; see, e.g., Çinlar [21], Chapter VI. Let $\{\theta_1, \ldots, \theta_J\}$ be a sample where $\theta_j \sim \mathcal{P}$, and as in Section 3.2 we observe samples $D_j = \{(X_{ij}, Y_{ij}) \sim G_j, i = 1, \ldots, N_j\}, j = 1, \ldots, J$, where $G_j$ stands for $G_{\theta_j}$. Given $\theta \in \Theta$, we consider the sample $D = \{(X_i, Y_i) \sim G_\theta, i = 1, \ldots, n\}$. For any function $f$ for which the conditional expectation $E_{G_\theta} f(D)$ of $f(D)$ given $G_\theta$ is well defined, we assume that so is $E_\mathcal{P} E_{G_\theta} f(D) = \int E_{G_\theta} f(D) \mathcal{P}(d\theta)$, where the outer expectation is over $\theta \sim \mathcal{P}$.

For any $G_\theta$ we define $\hat{\beta}_{\theta, n}$ to be the least squares estimator for a given sample $D$. If $G_\theta$ is sampled randomly from $\mathcal{P}$ then the population prediction error is defined as

$$ R_{pop}(n, p) := \int R_\theta(n, p) \mathcal{P}(d\theta) := \int E_{G_\theta} (Y - X' \hat{\beta}_{\theta, n})^2 \mathcal{P}(d\theta), $$

(22)

where the expectation $E_{G_\theta}$ is over $\hat{\beta}_{\theta, n}$ and $(X, Y) \sim G_\theta$ that are independent of $\hat{\beta}_{\theta, n}$. Let $\beta_\theta := \arg \min_\beta E_{G_\theta} (Y - X' \beta)^2$, $Q_\theta := E_{G_\theta} (XX')$, $\mathcal{W}_\theta := E_{G_\theta} (XX'c^2)$, and $\mathcal{V}_\theta := \mathcal{W}_\theta Q_\theta^{-1}$. As before, $R_{pop}(n, p)$ is approximated by

$$ AR_{pop}(n, p) := \int AR_\theta(n, p) \mathcal{P}(d\theta) := \int \left\{ E_{G_\theta} (Y - X' \beta_\theta)^2 + \frac{tr(\mathcal{V}_\theta)}{n} \right\} \mathcal{P}(d\theta), $$

(23)

where the latter integrand defines $AR_\theta(n, p)$ consistently with (7). The quantity $AR_\theta(n, p)$ whose estimation was already discussed, is now a random variable, since it depends on $G_\theta$ with $\theta \sim \mathcal{P}$; its expectation, given by (23), is the basis of our estimation of $R_{pop}(n, p)$ of (22). Lemma 3.4 below generalizes (9).
Lemma 3.4. Suppose that the conditions of Theorem 2.1 hold uniformly in $\theta \in \Theta$; that is, for each $k$, the $k$th moment with respect to $G_\theta$ of each entry of $X$ and $Y$ is bounded uniformly in $\theta$, and the entries of $(X_n'X_n/n)^{-1}$ have third moments with respect to $G_\theta$ that are bounded uniformly in $n$ and $\theta$. Then

$$R_{pop}(n,p) = AR_{pop}(n,p) + O(1/n^{3/2}).$$

The lemma clearly holds if $\Theta$ is finite, and in general it follows readily by the uniform boundedness of moments in $\theta$ and the proof of [7] given in the Appendix. Recall Lemma 2.2 where we showed that the moment conditions of Theorem 2.1 hold when $X$ is a mixture of normals and $\inf_{\Sigma \in \Xi} \lambda_{\min}(\Sigma) > 0$. For the moment bound of Lemma 3.4 to hold uniformly, it suffices that $\inf_{\Sigma \in \Xi} \lambda_{\min}(\Sigma) > 0$, where now the infimum is over all covariance matrices of all the mixing normals involved in all the distributions $G_\theta$ for all $\theta \in \Theta$. This technical assumption means that the covariates that are taken into account for the model selection are “bounded away” from multicollinearity. For discrete variables we redefine the prediction error by conditioning as in Section 2.4. Theorem 2.3 extends easily when we assume that all covariates are uniformly bounded in $\theta$, and that $\lambda_{\min}(\Xi) > c$ for some $c > 0$, for all $\theta$.

Lemma 3.4 suggests that a consistent estimator of $AR_{pop}(n,p)$ will lead to selection of an adequate model in the sense of Section 2.2 that is, a model that is as good as the model that minimizes $R_{pop}(n,p)$.

As in Section 3 we observe $D_j = \{(X_{ij},Y_{ij}) \sim G_j, i = 1,\ldots,N_j\} j = 1,\ldots,J$, where $G_j$ stands for $G_{\theta_j}$ where now $G_1,\ldots,G_J$ are sampled at random from $\mathcal{P}$. Recall the definition of $AR(n,p)$ in (17); here this quantity is considered random as it is a function of the sampled distributions $G_1,\ldots,G_J$. In order to generalize the consistency results of Theorem 3.1 to this case, we need to bound $AR_{pop}(n,p) - AR(n,p)$ as in the lemma below.

Lemma 3.5. Under the conditions of Lemma 3.4

$$AR_{pop}(n,p) - AR(n,p) = O_p(1/\sqrt{J})$$

uniformly in $n$. Moreover, for any fixed $n$, $\sqrt{J}(AR_{pop}(n,p) - AR(n,p))$ is asymptotically normal.

We now consider the population versions of the quantities defined in Section 3.2

$$p_{pop}^*(n) := \arg \min_{p} R_{pop}(n,p) = \arg \min_{p} \int E_{G_\theta}(Y - X^{(p)}Q_{\beta_\theta}^{(p)})^2 P(d\theta),$$

$$\pi_{pop}^*(n) := \arg \min_{p} AR_{pop}(n,p) = \arg \min_{p} \int \{E_{G_\theta}(Y - X^{(p)}Q_{\beta_\theta}^{(p)})^2 + \frac{tr(Y^{(p)}\beta_\theta^{(p)}\beta_\theta^{(p)T})}{n}\} P(d\theta),$$

$$p_{pop}^* := \arg \min_{p \in \mathcal{M}_{pop}} |p| \text{ where } \mathcal{M}_{pop} := \arg \min_{p} \int E_{G_\theta}(Y - X^{(p)}Q_{\beta_\theta}^{(p)})^2 P(d\theta).$$

Proposition 3.6 parallels Proposition 3.3; it shows consistency properties of $\hat{\pi}(n,N)$, as defined in (20) using (18). Below, the probability $P$ is obtained by first conditioning on $\theta_1,\ldots,\theta_J$, and
then unconditioning by taking expectation over $\theta_1, \ldots, \theta_{\mathcal{J}}$ with respect to the product measure $\mathcal{P}$. 

**Proposition 3.6.** Assume that the conditions of Lemma 3.4 hold, and in addition that $n/N_j, \lambda_{\max}(\nabla^p_{\theta})$, $1/\lambda_{\min}(\nabla^p_{\theta}), \lambda_{\max}(\nabla^p_{\theta}) \leq C$ for all $\theta$.

1. When $n$ is fixed,

$$\lim \inf_{N \to \infty} P \left( \hat{\pi}^*(n, N) \subseteq \pi_{\text{pop}}^*(n) \right) \geq 1 - \frac{K_C}{\mathcal{J}},$$

2. Letting $n, N \to \infty$,

$$\lim \inf_{n \to \infty, N \to \infty} P \left( \hat{\pi}^*(n, N) = p_{\text{pop}}^*(n) \right) \geq 1 - \frac{K_C}{\mathcal{J}},$$

where $K_C$ depends only on $C$.

The proof of Proposition 3.6 also shows that $p_{\text{pop}}^*$ is a singleton and $p_{\text{pop}}^*(n), \pi_{\text{pop}}^*(n)$ converge to it when $n \to \infty$.

4. **GENO**

4.1. **Definition of GENO**

Given a model $p$ with coefficients estimated by a sample of $n$ observations, we can say that it is equivalent to another model $q$ with $m$ observations if their expected prediction errors satisfy $R(m, q) = R(n, p)$. Using the approximations $AR(n, p)$ to $R(n, p)$ given in (7), (17), and (23) for each of the scenarios we consider, we define GENO by

$$\text{GENO}(n; p, q) := \left\{ m : AR(m, q) = AR(n, p) \right\}. \quad (25)$$

If $AR(m, q) > AR(n, p)$ for all $m$, then we set $\text{GENO}(n, p, q) = \infty$, indicating that model $p$ with $n$ observations is better than model $q$ with any number of observations. A direct calculation shows that for $\mathcal{J} = 1$ we have

$$\text{GENO}(n; p, q) = \operatorname{tr}(\nabla^q) \left\{ AR(n, p) - AR(n, q) + \frac{1}{n} \operatorname{tr}(\nabla^q) \right\}^{-1}.$$ 

For $\mathcal{J} > 1$ with $AR$ defined in (17) we have

$$\text{GENO}(n; p, q) = \left[ \frac{1}{\mathcal{J}} \sum_j \operatorname{tr}(\nabla^q_j) \right] \left\{ AR(n, p) - AR(n, q) + \frac{1}{\mathcal{J}n} \sum_j \operatorname{tr}(\nabla^q_j) \right\}^{-1}. $$

For the case of (23), $j$ is replaced by $\theta$, and the averages are replaced by integrals $\mathcal{P}(d\theta)$.

$\text{GENO}(n; p, q) = m$ means that model $p$ with $n$ observations is equivalent, in terms of expected prediction error, to model $q$ with $m$ observations. Note that the larger $\text{GENO}(n; p, q)$ is, the better
model $p$ (with $n$ observations) is relative to model $q$. For each model $p$ and sample size $n$, we define

$$\text{GENO}(n, p) = \min_r \text{GENO}(n; p, r).$$

(26)

It follows that $\text{GENO}(n, p) \leq n$, where equality means that $p$ is the best model for $n$ observations, as no other model can achieve the same prediction error with fewer observations. On the other hand, $\text{GENO}(n, p) = m < n$ means that there is a model that achieves, with $m < n$ observations, the same prediction error as $p$ with $n$ observations. Thus, a small value of $\text{GENO}(n, p)$ suggests considering another model. By the monotonicity of $\text{AR}(n, p)$ in $n$, if the inequality $\text{GENO}(n; p, r) \geq \text{GENO}(n; q, r)$ holds for some model $r$, then it holds for any model. This readily implies

$$\text{AR}(n, p) \leq \text{AR}(n, q) \iff \text{GENO}(n; p, r) \geq \text{GENO}(n; q, r) \text{ for all } r \iff \text{GENO}(n, p) \geq \text{GENO}(n, q).$$

(27)

4.2. Estimation of GENO

We apply (13) of Theorem 2.1, which shows the consistency of $C(p)(n, N)$ for fixed $n$ as $N \to \infty$, to estimate $\text{AR}(m, p)$ and $\text{AR}(n, q)$ in (25) by $C(p)(m, N)$ and $C(q)(n, N)$ as defined (for the model $p$) in (8), and define

$$\hat{\text{GENO}}(n; p, q) := \{ m : C(q)(m, N) = C(p)(n, N) \}.$$

To avoid cumbersome notation we suppress $N$ in the definition. Using (8) we obtain, as before,

$$\hat{\text{GENO}}(n; p, q) = \text{tr}(\hat{\mathbb{V}}^{(q)}_{N}) \left\{ C(p)(n, N) - C(q)(n, N) + \frac{\text{tr}(\hat{\mathbb{V}}^{(q)}_{N})}{n} \right\}^{-1},$$

(28)

setting $\hat{\text{GENO}}(n, k, \ell) = \infty$ if the expression in curly brackets is negative or zero.

In the case of $J$ samples of Section 3.1, or in the population case of Section 3.3, the above expression remains unchanged except that now $C(p)(n, N)$ is replaced by $C(p)(n, N)$ defined in (18), and $\hat{\mathbb{V}}^{(q)}_{N}$ is replaced by $\frac{1}{J} \sum_{j} \text{tr}(\hat{\mathbb{V}}^{(q)}_{N,j})$. We can also define the estimator of (28) in terms of the bias-corrected $C(p)(n, N)$ of (19), while bias-correcting also $\text{tr}(\hat{\mathbb{V}}^{(q)}_{N})$ as an estimator of $\text{tr}(\mathbb{V}^{(q)})$. This is done in estimating GENO in Section 6.3. The results below hold in the same way for all these cases. Similarly to (26), we define the statistic

$$\text{GENO}(n, p) := \min_r \hat{\text{GENO}}(n; p, r),$$

which is an estimate the minimal number of observations required by the best competing model to achieve the same prediction error as model $p$ with sample size $n$.

As in (27), we have

$$C(p)(n, N) < C(q)(n, N) \iff \hat{\text{GENO}}(n; p, r) \geq \hat{\text{GENO}}(n; q, r) \text{ for all } r \iff \text{GENO}(n, p) \geq \text{GENO}(n, q).$$

The next proposition follows from (13) by applying the $\delta$-method to the inverse function in (28). In particular, it shows the consistency of $\hat{\text{GENO}}(n; p, q)$ for fixed $n$ as $N \to \infty$. 

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Proposition 4.1. Under the conditions of Theorem 3.1 (which include the case \( J = 1 \)), we have for any fixed \( n \)

\[
\sqrt{N_1} (\text{GENO}(n; p, q) - \text{GENO}(n; p, q)) \overset{D}{\to} N(0, \eta^2) \text{ as } N \to \infty,
\]

for some \( \eta^2 > 0 \).

The variance \( \eta^2 \) is not given explicitly since it is too complicated to be useful, and it can be computed by the bootstrap. See Theorem 3.1 and the ensuing comment, which show that (under certain conditions) the variance decreases at a rate of \( 1/J \).

5. Simulations

In this section we evaluate \( R(n, p) \), its approximation \( AR(n, p) \), and its estimation using \( C(p) \) by simulations. We start with the single sample \( (J = 1) \) and then we consider the case of several samples. This simple example demonstrates the well-known difficulty involved in model selection for a single given dataset with methods such as Mallows \( C_p \), AIC, BIC, and our version of \( C(p) \).

In Section 5.2 we compare the case of one sample to the task of choosing a common model for successful prediction on the average when we have data from several samples.

5.1. A single sample

Suppose that the distribution of \((X, Y)\) for \( X \in \mathbb{R}^{10} \) is given by

\[
Y = b_0 + b_1X_1 + \ldots + b_{10}X_{10} + a(X_1^2 - 1) + \sigma\varepsilon,
\]

with \( X_1, \ldots, X_{10}, \varepsilon \sim^i d N(0, 1) \). There are \( 2^{10} \) possible submodels; for simplicity, we focus on three models consisting of the subsets of covariates \( p_1 = \{1, X_1\}, \ p_2 = \{1, X_1, \ldots, X_5\}, \) and \( p_3 = \{1, X_1, \ldots, X_{10}\}, \) or more explicitly, we have model \( p_1: Y = \beta_0 + \beta_1X_1 + e \), model \( p_2: Y = \beta_0 + \beta_1X_1 + \ldots \beta_5X_5 + e \), model \( p_3: Y = \beta_0 + \beta_1X_1 + \ldots \beta_{10}X_{10} + e \). All models are wrong since the residual \( e \) includes the nonlinear term \( X_1^2 - 1 \). By the orthogonality of the variables in (29), the projection parameters \( \beta_k \) are equal to \( b_k \) for all these models; see (2). This is used to compute the first part of \( AR(n, p_\ell) \) for \( \ell = 1, 2, 3 \), and since in this case \( Q = I \), it is also easy to compute \( tr(V) \) for each model. We obtain

\[
AR(n, p_1) = \sum_{k=2}^{10} b_k^2 + 2\alpha^2 + \sigma^2 + \frac{2(\sum_{k=2}^{10} b_k^2 + \sigma^2) + 12\alpha^2}{n},
\]

\[
AR(n, p_2) = \sum_{k=6}^{10} b_k^2 + 2\alpha^2 + \sigma^2 + \frac{6(\sum_{k=6}^{10} b_k^2 + \sigma^2) + 20\alpha^2}{n}, \text{ and } AR(n, p_3) = 2\alpha^2 + \sigma^2 + \frac{11\sigma^2 + 30\alpha^2}{n};
\]

(30)
notice that the above functions do not depend on $b_0, b_1$. For a concrete example, we set in (29) 
$(b_0, b_1) = (0, 0), (b_2, \ldots, b_5) = (1.5, \ldots, 1.5), (b_6, \ldots, b_{10}) = (1, \ldots, 1), a = 3$, and $\sigma = 10$. Figure
1 plots $R(n, p_\ell)$ of (5) (in the solid line) and $AR(n, p_\ell)$ of (7) (in the dashed line), $\ell = 1, 2, 3$, as 
functions of $n$ for the above parameters. We estimated $R(n, p_\ell)$, where $\ell = 1, 2, 3$, by a simulation 
based on $10^4$ repetitions. For small $n$, $R(n, p_2)$ and $R(n, p_3)$ differ from $AR(n, p_2)$ and $AR(n, p_3)$ 
(more so for $p_3$), and the approximation improves as $n$ increases. For $n$ smaller than about 60, 
model $p_1$ has a smaller prediction error; for $n$ between 60 and 130, model $p_2$ is better, and for 
larger $n$, $p_3$ is optimal (among the three models). This holds approximately for both $R$ and $AR$. 
This makes sense as models with fewer parameters have a smaller prediction error for small $n$.

Consider GENO as defined in (25). Careful inspection of Figure 1 shows, for example, that 
GENO(53; $p_1, p_2$) = 53, which means that to achieve the same prediction error as model $p_1$ with 
$n = 53$ observations (the value of $n$ where the dashed black line and red the line intersect), model $p_2$ 
requires the same number of observations. Also, GENO(60; $p_2, p_1$) = 75, and therefore, to achieve 
the same prediction error as model $p_2$ with $n = 60$, model $p_1$ would require 75 observations (the 
value of $n$ where the dashed black line has the same level as the red line at 60). Since the decrease 
of $AR(n, p_1)$ (the black line) in $n$ is slow, a small increase in $n$, will result in a much larger value 
of $GENO(n; p_2, p_1)$; for example, GENO(65; $p_2, p_1$) = 97. As mentioned before, GENO allows 
the statistician to compare the cost of additional observations to the cost of measuring additional 
variables, which may be expensive, or harmful, such as an invasive medical procedure or imaging 
that involves radiation.

We now discuss estimation of the prediction error using $C^{(p)}(n, N)$ based on a single sample of
size $N = 40$. Figure 2 plots $R(n, p_1) - R(n, p_2)$ (solid line), $AR(n, p_1) - AR(n, p_2)$ (dashed line), and boxplots of the estimators $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$ on the left-hand side, and the jackknife bias-corrected version $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$ on the right-hand side, based on $10^3$ simulations for each $n = 30, 40, \ldots, 250$. Their means are given by circles. We see that the jackknife performs very well in correcting the bias of $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$ as an estimator of $AR(n, p_1) - AR(n, p_2)$; see the discussion following (8). Recall that the bias itself and the correction decrease in $n$. The mean of the difference $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$ and $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$ equals 0 at about $n = 30$ and $n = 60$, respectively; thus the jackknife leads to correct selection on average since it is optimal to select model $p_1$ for about $n \leq 60$.

![Figure 2](image1.png)

(a) Boxplots of $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$

(b) Boxplots of $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$

Figure 2: (a) Boxplots of the simulation results of $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$, where ◦ (circle) denotes the mean, and $R(n, p_1) - R(n, p_2)$ (respectively, $AR(n, p_1) - AR(n, p_2)$) is a solid (respectively, dashed) line. (b) Same as (a) for jackknifed version $C^{(p_1)}(n, N) - C^{(p_2)}(n, N)$.

Figure 3 depicts the probability of selecting model $p_1$ as a function of $n$, using $C^{(p)}(n, N)$ and the jackknifed $C^{(p)}(n, N)$. The bias correction increases the probability of selecting model $p_1$ and $p_2$. This improves the selection for small or moderate values of $n$. For large $n$ the difference between the three models (see Figure 1) is not large, and this is reflected by the selection probabilities. For the problem of selecting a common model for $J$ samples, the bias correction becomes significant, as demonstrated next.

5.2. Multiple samples

We now consider the multiple samples case. Suppose that $G_\theta$ is given by model (29) with $(b_{\theta,0}, b_{\theta,1}) = (0, 0)$, $b_{\theta,k} \sim N(1.5, 0.1^2)$, $k = 2, \ldots, 5$, $b_{\theta,k} \sim N(1, 0.1^2)$, $k = 6, \ldots, 10$, $a_\theta = 3$, and
\( \sigma_\theta = 10 \), where all the above random variables are independent (thus determining the distribution \( P \) of Section 3.3), and then fixed throughout this section. The number of regression samples is \( J = 200 \), and \( N_j = 40, 100, 150, \) and \( 250 \) for \( 1 \leq j \leq 50, 51 \leq j \leq 100, 101 \leq j \leq 150, \) and \( 151 \leq j \leq 200 \), respectively. In the case of observing all regressions (see Section 3.1, Equation (16)), we wish to estimate \( R(n, p) \), whereas in the case of observing a sample of regressions from the distribution \( P \) (see Section 3.3, Equation (22)), the relevant quantity is \( R_{pop}(n, p) \). Computing the latter quantity is difficult, and instead we use the approximation \( R(n, p) \), which is justified by the law of large numbers and the central limit theorem (see Lemma 3.5). Thus we now focus on estimating \( R(n, p) \) and selecting according to its estimates. The plot of \( R(n, p) \) for \( p = p_1, p_2, p_3 \) is similar to Figure 1 and therefore is not presented here.

Figure 4 parallels Figure 2, where now in the \( J \)-sample case, \( C^{(p)}(n, N) \) and \( C^{(p)}(n, N) \) replace \( C^{(p)}(n, N) \), and \( C^{(p)}(n, N) \), respectively; see (18) and (19). Again we see that the jackknife bias correction works well. Here the variances of the estimates are much smaller, indicating that the several samples formulation allows for better estimates and model selection procedures as predicted by theory. The \( y \)-axis scale varies between Figures 4 and 2 in a way that undermines their difference.

Figure 5 plots the selection probabilities as a function of \( n \). Unlike the case \( J = 1 \) (see Figure 3), model \( p_1 \) (respectively, model \( p_2 \)) is selected with high probability for small \( n \) (respectively, large \( n \)). Recall that it is optimal to select model \( p_1, p_2, \) and \( p_3 \) when \( n \leq 60, 60 \leq 130, \) and \( n \geq 130, \) respectively. Selecting according to \( C^{(p)}(n, N) \) leads to favoring model \( p_3 \) when \( n \) is greater than 70 (instead of 130) and \( C^{(p)}(n, N) \) corrects this bias. From Figure 3 we see that if we select a regression model for a single regression \( (J = 1) \) with \( N = 40 \) observations, which we also
use for prediction (that is, \( n = 40 \)), then we select model \( p_1 \) (the better model) with probability \( \approx 0.5 \). Figure 5 shows that if we use the \( J = 200 \) regressions as described above for the model selection, then for \( n = 40 \) we choose model \( p_1 \) with probability 1. When \( N = n = 150 \), for which \( p_3 \) is the best model, we see that on the basis of a single regression we would choose model \( p_3 \) with probability \( \approx 0.55 \), while using the \( J = 200 \) regressions leads to choosing model \( p_3 \) with probability close to 1. Thus, we achieve almost correct model selection when using the \( J = 200 \) samples for each of the regressions (see Figure 1). Clearly, the probability of making a correct selection depends on the number of samples \( J \), the similarity among the \( J \) models, the noise level in the models, and the sample size \( n \).

Figure 4: Same plots as in Figure 2 when there are \( J = 200 \) samples.

Figure 5: Same plots as in Figure 3 when there are \( J = 200 \) samples.
6. Prediction of duration of medical examination

In this section we analyze a dataset of outpatients’ hospital visits. Different models are considered in order to predict the actual appointments’ duration as opposed to the planned duration.

6.1. Description of the data

The dataset analyzed is taken from the SEE Lab at the Technion. It consists of information on 140,924 hospital visits that took place in a certain US hospital for about two years between 2013 and 2015. For each visit, both the planned time and the actual time are reported. The goal was to provide a more accurate prediction of the actual duration than the planned one. In this dataset there is information 44,516 patients and 258 doctors, out of whom 34 doctors had fewer than 50 visits. We shall focus on the rest, which corresponds to 99.5% of all visits. The regression coefficients will differ between doctors, and the goal is to select one common subset of covariates (for each $n$) for all doctors for prediction of visit durations.

The distribution of the planned duration is given in Table 1 and Figure 6 plots the estimated density (a normal kernel estimate using the R command “density”) of the actual durations for the time slots of 15, 30, and 60 minutes. Actual durations are obtained by a real-time location system (RTLS). The means are 16.7, 21.3, and 41.2, respectively.

Table 1: The distribution of the planned duration.

<table>
<thead>
<tr>
<th>minutes</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>other</th>
</tr>
</thead>
<tbody>
<tr>
<td>percentage</td>
<td>29.8%</td>
<td>52.6%</td>
<td>1.8%</td>
<td>15.5%</td>
<td>0.3%</td>
</tr>
</tbody>
</table>

Figure 6: Estimated density of the actual duration for the time slots of 15, 30, and 60 minutes. The vertical dashed lines are at 15, 30, and 60 minutes.
6.2. A regression model

The original dataset contains a large number of covariates, of which many did not seem to have any predictive power relative to visit durations. For simplicity, we focus on a small number of covariates that seem most relevant. We aim to predict actual duration, using the following covariates:

- \texttt{duration\_planned} = the planned duration of the visit in minutes.
- \texttt{duration\_planned}^2 = the planned duration in minutes of the visit, squared.
- \texttt{last} = the planned minus the actual duration of the previous visit of the same patient (taken to be 0 for the first visit of the patient).
- \texttt{hour\_end} = whether the exam is planned to end on the hour. It turns out that these kinds of visits tend to be somewhat longer.
- \texttt{type} = there are two types of examinations: consultation/examination only, or the above plus treatment. In either case, only the first part counts as duration.

The output of the “\texttt{lm}” command in R applied to the whole dataset (ignoring the doctors’ index) is given in Figure 7. All of the above covariates besides “\texttt{type}” are statistically significant; however, the standard error of the residuals is 15.33, and $R^2 = 0.227$, suggesting that the prediction error is quite large.

![Figure 7: The output of the R “\texttt{lm}” procedure for the global linear model.](image)

6.3. $\mathcal{C}^{(p)}$ and model selection

In our notation each doctor is indexed by $j$, and $N_j$ is the number of visits to doctor $j$ in the dataset. We demonstrate our approach by focusing on four candidate models that seemed the most
relevant submodels of the five covariates (all models included the intercept term). These models are $p_1$ – the model with the covariates: duration_planned, duration_planned_2; $p_2$ – the model with the same covariates as in $p_1$ and additionally, the variable “last”; $p_3$ – the model with the same covariates of $p_2$ and additionally, the variable “type”; and $p_4$ – the full model. For certain submodels estimation is possible only for a subset of the doctors since $X_j^{(p)} X_j^{(p)}'$ is not always invertible. Therefore $J$ varies between the models. For the models $p_1$ and $p_2$, invertibility held for 96 doctors and for the models $p_3$ and $p_4$, the corresponding number is 95, and so for these models $J = 95$ or $J = 96$. In this case, $C^{(p)}(n, N)$ is based only on this subset.

Figure 8 plots $C^{(p)}(n, N)$ for $p = p_1, p_2, p_3, p_4$ and $n$ between 50 and 500. For $n$ smaller than approximately 80, model $p_1$ is the best among the candidate models; for $n$ between 80 and 450, $p_2$ has a smaller $C^{(p)}$; and for larger $n$, $p_3$ is the best, but $p_2$ is very close. In terms of GENO, we have, for example, that for $n = 50$, $\hat{\text{GENO}}(n, p_1, q)$ for $q = p_2, p_3, p_4$ equals 54, 63, 73, respectively. The latter number means that model $p_4$ (the full model) would require 73 observations to achieve the same prediction error as model $p_1$ with $n = 50$ observations. Also, $\hat{\text{GENO}}(200, p_2, p_1) = 370$; if one considers using only the planned duration ($p_1$) or using model $p_2$, that is, adding the variable “last” with the information on the last visit, which may not be available for some patients, then the estimated prediction error by the model $p_2$ with $n = 200$ observations can be achieved without knowing “last” by $p_1$, with $n = 370$. It is then left to the user to decide whether to invest in measuring “last” or in using a larger sample.

![Figure 8: A plot of $C^{(p)}(n, N)$ for $p = p_1, p_2, p_3, p_4$ and $n = 50, 55, \ldots, 500$.](image)

Table 2 reports $C^{(p)}(n, N)$ for $p = p_1, p_2, p_3, p_4$ and for different sample sizes $n$. Standard deviations estimated by the bootstrap, and cross-validation estimates of $R(n, p)$, are also provided.
The latter estimates are computed only for $j$’s where $N_j > n$. For each such $j$, the data were split at random into a training set with $n$ observations, and a testing set of size $N_j - n$. The estimates $\hat{\beta}_{j,n}$ are based on the training set and the prediction error is evaluated using the testing set. This procedure was repeated 1,000 times and the average prediction error is reported. The cross-validation estimates are mostly within one standard error of the $\text{CCC}^{(p)}$ values, and the two approaches lead to selection of the same models.

Table 3 reports the values of $\text{CCC}^{(p)}(n,N) - \text{CCC}^{(q)}(n,N)$ together with a bootstrap estimate of the standard deviation for different values of $n$ and various pairs of models. Also the differences of the corresponding cross-validation estimates are given. The standard deviations of Table 3 are much smaller than those of Table 2. This is consistent with our theoretical results that comparison of two similar models leads to a small estimation error (see the discussion after Theorem 2.1).

Table 2: $\text{CCC}^{(p)}(n,N)$ for $p = p_1, p_2, p_3, p_4$ and for $n = 50, 150, 500$. Bootstrap standard deviations (SD) and cross-validation (CV) estimates are also provided.

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<thead>
<tr>
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<th>Model $p_1$</th>
<th>Model $p_2$</th>
<th>Model $p_3$</th>
<th>Model $p_4$</th>
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</thead>
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<tr>
<td></td>
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<td>CV</td>
<td>$\text{CCC}^{(p_2)}$ (SD)</td>
<td>CV</td>
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<tr>
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<tr>
<td>150</td>
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<tr>
<td>500</td>
<td>178.1 (3.3)</td>
<td>183.0</td>
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<td>181.3</td>
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</table>

Table 3: The values of $\text{CCC}^{(p)}(n,N) - \text{CCC}^{(q)}(n,N)$ for $n = 50, 150, 500$. Bootstrap standard deviations (SD) and cross-validation (CV) estimates are also provided. Boldface numbers indicate differences that are significantly (more than two SD’s) non-zero.

<table>
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<tr>
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<td>p</td>
<td>q</td>
<td>$\text{CCC}^{(p)} - \text{CCC}^{(q)}$ (SD)</td>
<td>CV</td>
<td>$\text{CCC}^{(p)} - \text{CCC}^{(q)}$ (SD)</td>
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<tr>
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<td>-1.0 (0.2)</td>
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**Acknowledgment:** We are grateful to Avishai Mandelbaum for providing access to the SEE Lab dataset we analyzed, and to Ella Nadjarov for creating the files we needed.
References


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7. Appendix A: Proofs

Recall that we use $p$ to denote a subset of the covariates, which we refer to as a model, and also to denote the subset’s size.

**Proof of Theorem 2.1** We first prove (9). For $\hat{\beta}_n$ computed from a sample $D = \{(X_i, Y_i) : i = 1, \ldots, n\}$, and a pair of new observations from the same distribution $(X, Y)$, independent of $D$, we have

$$E(Y - X'\hat{\beta}_n)^2 = E(Y - X'\beta)^2 + E[X'(\hat{\beta}_n - \beta)]^2 - 2E[(Y - X'\beta)X'(\hat{\beta}_n - \beta)]$$

$$= E(Y - X'\beta)^2 + E[X'(\hat{\beta}_n - \beta)]^2, \quad (31)$$

where the last term in the first line of (31) vanishes since $E[(Y - X'\beta)X] = E[\epsilon X] = 0$ and $Y$ and $X$ are independent of $\hat{\beta}_n$. This argument holds also if we condition on $G_n$ (see (14), and Theorem 2.3). By (31) we have that

$$n[R(n, p) - AR(n, p)] = E[X'\sqrt{n}(\hat{\beta}_n - \beta)]^2 - tr(V). \quad (32)$$

Using independence of $X$ and $\hat{\beta}_n$ again we have

$$E[X'\sqrt{n}(\hat{\beta}_n - \beta)^2] = tr\{E[XX']E[n(\hat{\beta}_n - \beta)(\hat{\beta}_n - \beta)']\} = tr\{QE[n(\hat{\beta}_n - \beta)(\hat{\beta}_n - \beta)']\}. \quad \text{(33)}$$

For the proof of Theorem 2.3 the expectations should be conditioned on the set $G_n$, whose probability is large, and the conditioning does not affect the rates we obtain.

By Equation (7.3) of Hansen [9],

$$\sqrt{n}(\hat{\beta}_n - \beta) = \hat{\beta}_n^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i e_i = \hat{\beta}_n^{-1} U_n.$$

Since $E\{tr(U_nU'_nQ^{-1})\} = tr(V)$ we rewrite the right-hand side of (32) as

$$E\{tr(Q\hat{\beta}_n^{-1} U_n U'_n \hat{\beta}_n^{-1}) - tr(QQ^{-1}U_n U'_n QQ^{-1})\} = tr(QE\{\hat{\beta}_n^{-1} U_n U'_n \hat{\beta}_n^{-1} - QQ^{-1}U_n U'_n QQ^{-1}\}).$$

In order to prove (9) we now show that the latter expectation is of order $O(1/n^{1/2})$. To this end, notice that

$$\hat{\beta}_n^{-1} U_n U'_n \hat{\beta}_n^{-1} - QQ^{-1}U_n U'_n QQ^{-1} = (\hat{\beta}_n^{-1} - QQ^{-1})U_n U'_n \hat{\beta}_n^{-1} + QQ^{-1}U_n U'_n (\hat{\beta}_n^{-1} - QQ^{-1}).$$

We now deal with the first term on the right-hand side above, the other term being similar, and simpler. We have

$$\hat{\beta}_n^{-1} - QQ^{-1} = QQ^{-1}(Q - \hat{\beta}_n) \hat{\beta}_n^{-1} \quad \text{(33)}$$

and therefore we consider the trace of $(Q - \hat{\beta}_n) \hat{\beta}_n^{-1} U_n U'_n \hat{\beta}_n^{-1}$. This is a product of random matrices of the form $ABCD$ where $A = Q - \hat{\beta}_n$, $B = D = \hat{\beta}_n^{-1}$, and $C = U_n U'_n$. The trace
is a sum of products of entries from all the matrices appearing in the product. Different choices of powers can be made, but for simplicity we use Hölder’s inequality in the form $E|abcd| \leq (Ea^{12})^{1/12}(Eb^{3})^{1/3}(Ec^{4})^{1/4}(Ed^{3})^{1/3}$. Here $a$ is an entry from $A$, $b$ an entry from from $B$, etc., and the triangle inequality can then be used to bound the sum comprising the trace.

For each element $j,k$, $Q - \hat{Q}_n$, we have

$$E(\hat{Q}_n)_{j,k}^2 = E \left( \frac{1}{n} \sum_{i=1}^{n} [E(X_jX_k) - X_{ij}X_{ik}] \right)^{12}.$$ 

The summands $E(X_jX_k) - X_{ij}X_{ik}$ have zero expectation, and expanding $(\hat{Q}_n)_{j,k}^2$ we see that the number of nonvanishing terms when expectation is taken is of order $n^6$, and all these terms are bounded by our assumptions. Therefore, $E(\hat{Q}_n)_{j,k}^2$ is of order $1/\sqrt{n}$. (Actually, it is easy to see that 24 bounded moments suffice for this argument, and also for bounding the remaining terms, and 24 can be somewhat reduced by a better but more cumbersome choice of the powers in Hölder’s inequality.) A similar computation for the matrix $C$ shows that the required moments of its entries are bounded. The rest of the terms are bounded by our assumptions. Now follows.

The proof required the bounded third power of $B = \hat{Q}_n$, which means that with a mixture of normals we need $n > p + 5$. See the Proof of Lemma 2.2.

We now show (10). The definitions of $AR(n,p)$ and $C(p)(n,N)$ imply that

$$AR(n,p) - C(p)(n,N) = E(Y - X'\hat{\beta})^2 - \frac{1}{N}||Y_N - X_N\beta_N||^2 + \frac{1}{n} \left\{ tr(V) - tr(\hat{V}_N) \right\} - \frac{tr(\hat{V}_N)}{N}. \quad (34)$$

Starting with the second term on the right-hand side of (34), we have

$$||Y_N - X_N\beta||^2 = ||Y_N - X_N\hat{\beta}_N||^2 + ||X_N(\hat{\beta}_N - \beta)||^2 - 2(Y_N - X_N\hat{\beta}_N)'(X_N(\hat{\beta}_N - \beta)),$$

where the last term vanishes since $X_N'(Y_N - X_N\hat{\beta}_N) = 0$. Hence,

$$E(Y - X'\beta)^2 - \frac{1}{N}||Y_N - X_N\hat{\beta}_N||^2 = E(Y - X'\beta)^2 - \frac{1}{N}||Y_N - X_N\beta||^2 + \frac{1}{N}||X_N(\hat{\beta}_N - \beta)||^2. \quad (35)$$

Recall the notation $U_N = X'_N e_N / \sqrt{N}$. Since by Equation (7.3) of Hansen [9] $N^{-1/2}(\hat{\beta}_N - \beta) = N^{-1/2}(X'_N X_N)^{-1}X'_N e_N = (X'_N X_N)^{-1}U_N$, we have

$$\frac{1}{N}||\hat{\beta}_N - \beta||^2 = \frac{1}{N} tr\{U_N U'_N (X'_N X_N)^{-1} - (X'_N X_N)^{-1}U_N \} = \frac{1}{N} tr\{U_N U'_N (X'_N X_N/N)^{-1} - (X'_N X_N/N)^{-1}U_N \} + o_p(1/N),$$

where the last equality holds true since $(X'_N X_N/N)^{-1} - Q^{-1} = o_p(1)$. We have

$$\frac{1}{N}||\hat{\beta}_N - \beta||^2 = \frac{1}{N} \left[ tr(U_N U'_N Q^{-1}) - tr(WQ^{-1}) + tr(WQ^{-1}) \right] + o_p(1/N)\]

$$= \frac{1}{N} \left[ tr(U_N U'_N Q^{-1}) - tr(V) + tr(\hat{V}_N) \right] + o_p(1/N), \quad (36)$$
where for the last equality it suffices that \( \hat{Q}_N \) and \( \hat{\mathbb{W}}_N \) are consistent estimates of \( Q \) and \( \mathbb{W} \), and therefore \( \text{tr}(\hat{V}_N) \) is consistent for \( \text{tr}(\mathbb{V}) = \text{tr}(\mathbb{W}Q^{-1}) \). Equations (34), (35), and (36) imply (10).

Next we show that \( \sqrt{N}\{\text{tr}(\hat{V}_N) - \text{tr}(\mathbb{V})\} \) is asymptotically normal starting with the asymptotic normality of \( \sqrt{N}(\hat{\mathbb{W}}_N - \mathbb{W}) \). We have,

\[
\hat{\mathbb{W}}_N = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' e_i^2 = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' (Y_i - X_i' \hat{\beta}_N)^2 \\
= \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' (Y_i - X_i' \beta - \{X_i' \hat{\beta}_N - X_i' \beta\})^2 = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' (Y_i - X_i' \beta)^2 \\
- \frac{2}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' (Y_i - X_i' \beta) (X_i' \hat{\beta}_N - \beta) + \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' (\hat{\beta}_N - \beta)^2 =: A - B + C, \quad (37)
\]
respectively. Starting with the first term we have

\[
\sqrt{N}(A - \mathbb{W}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{n} (\mathbf{x}_i \mathbf{x}_i' e_i^2 - E[\mathbf{x} \mathbf{x}' e^2])
\]
which is asymptotically normal. Now \( B \) is obtained by multiplying \( \frac{2}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' (Y_i - X_i' \beta) \otimes X_i' \)
(which converges to a matrix of constants by the law of large numbers) by \( I_p \otimes (\hat{\beta}_N - \beta) \), where \( I_p \) is the identity matrix of order \( p \), and \( \otimes \) is Kronecker’s product. By Equation (7.3) of Hansen [9],

\[
\sqrt{N}(\hat{\beta}_N - \beta) = \hat{Q}_N^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{x}_i e_i.
\]

It follows that the term \( C \) in (37) is \( O_p(\frac{1}{N}) \) by the law of large numbers and the fact that \( N(\hat{\beta}_N - \beta)_{ij}(\hat{\beta}_N - \beta)_{jk} = O_p(1) \). Summing up,

\[
\sqrt{N}(\hat{\mathbb{W}}_N - \mathbb{W}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{n} (\mathbf{x}_i \mathbf{x}_i' e_i^2 - E[\mathbf{x} \mathbf{x}' e^2]) - B_N[I_p \otimes (\hat{Q}_N^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{n} \mathbf{x}_i e_i)] + O_p(1/N),
\]
where \( B_N \) is the matrix \( \frac{2}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i' (Y_i - X_i' \beta) \otimes X_i' \). Condition (i) implies that the second moment of \( \mathbf{x}_i \mathbf{x}_i' e_i^2 \) is finite. Hence, by the central limit theorem,

\[
\frac{1}{\sqrt{N}} \left( \sum_{i=1}^{n} \{\mathbf{x}_i \mathbf{x}_i' e_i^2 - E[\mathbf{x} \mathbf{x}' e^2]\}, \sum_{i=1}^{n} \mathbf{x}_i e_i \right)
\]
is jointly asymptotically normal, and since \( A_N \) and \( \hat{Q}_N \) converge to a matrix of constants, a version of Slutsky’s theorem implies that \( \sqrt{N}(\hat{\mathbb{W}}_N - \mathbb{W}) \) is asymptotically normal.

Another application of Slutsky’s theorem implies that \( \sqrt{N}(\hat{\mathbb{W}}_N \hat{Q}_N^{-1} - \mathbb{W}Q^{-1}) \) is asymptotically normal (since \( \hat{Q}_N \to Q \) in probability) and therefore so is

\[
\sqrt{N} \left\{ \text{tr}(\hat{\mathbb{W}}_N \hat{Q}_N^{-1}) - \text{tr}(\mathbb{W}Q^{-1}) \right\} = \sqrt{N} \left\{ \text{tr}(\hat{V}_N) - \text{tr}(\mathbb{V}) \right\}.
\]

It follows that \( \text{tr}(\hat{V}_N) - \text{tr}(\mathbb{V}) = O_p(1/\sqrt{N}) \) and it is asymptotically normal.
Similar to previous arguments, the random variables in \[38\] and the first part of \(E_N\) are jointly asymptotically normal and a version of Slutsky’s theorem that allows us to ignore the term \(O_p(1/N)\) together with \[10\] and \[11\], implies the last statement of Theorem 2.1 about the asymptotic normality of \(\sqrt{N}(C(p)(n,N) - AR(n,p))\).

**Proof of Lemma 2.2** Let \(\tilde{X}\) denote the \(n \times (p-1)\) matrix defined as \(X_n\) but without the first column of 1’s. We suppress \(n\) here and in the following notation. Let \(\tilde{X} := \tilde{X}/n\), where \(1\) is an \(n\)-column vector of 1’s, that is, \(\tilde{X}\) is the \((p-1)\)-column of covariates means, and let \(S := \tilde{X}/X - X\). Then by Horn and Johnson [10], page 25,

\[
(\tilde{X}X/n)^{-1} = \begin{bmatrix}
1 + \tilde{X}S^{-1}\tilde{X} & -\tilde{X}S^{-1} \\
-S^{-1}\tilde{X} & S^{-1}
\end{bmatrix}.
\]

Let \(\tilde{X}\) denote a \((p-1)\)-vector of the covariates without the first 1, and assume first that \(\tilde{X} \sim N(\mu, \Sigma)\). Then \(nS \sim Wishart_{p-1}(\Sigma, n-1)\), and \(\tilde{X}\) and \(S\) are independent. The third moments of \(S^{-1}\) are uniformly (in \(n\)) bounded by \(C \max(1, 1/\lambda_{\min}(\Sigma)^3)\) for some \(C > 0\), provided \(n-p-5 > 0\) by Theorem 4.1 in von Rosen [15]. If \(\tilde{X}\) is distributed according to a mixture of normals, the assumption in the lemma that all these normal distributions have \(\lambda_{\min}(\Sigma) > c > 0\) implies the uniform boundedness for the mixture.

**Lemma 2.3.** For \(X_n \in G_n\) (see [14]) the matrix \((X_n'X_n/n)^{-1}\) exists and all its entries are bounded uniformly in \(n\). Moreover, if the components of \(X\) are bounded, then for some \(a < 1\) we have, \(P(G_n) > 1 - a^{n\lambda_{\min}(Q)}\), which converge to 1 at an exponential rate in \(n\).

**Proof.** When \(X_n \in G_n\), then \(\lambda_{\min}(X_n'X_n/n) \geq \frac{1}{n} \lambda_{\min}(Q) > 0\), and therefore \((X_n'X_n/n)^{-1}\) exists. Since the entries of a positive semi-definite matrix are bounded by the maximal eigenvalue, all entries of \((X_n'X_n/n)^{-1}\) are bounded in this case by \(2/\lambda_{\min}(Q)\). For the moreover part, notice that when the elements of \(X\) are bounded then so is \(\lambda_{\max}(X_n'X_n/n)\). By Tropp [21], Theorem 1.1, \(P(G_n) \leq a^{n\lambda_{\min}(Q)}\) for some \(a < 1\).

**Proof of Theorem 2.3** Conditionally on \(G_n\) the arguments in the proof of Theorem 2.1 continue to apply with obvious modifications. Lemma 2.3 provides an exponentially small bound on \(P(B_n)\), where \(B_n = G_n^c\), adding a term \(O_p(e^{-cn})\) in Equation [9] and \(O_p(e^{-cN})\) to [10] and [12] (ii) for some \(c > 0\). Clearly, these terms do not affect the results.

**Proof of Lemma 2.4** For Part (i), set \(p(n) \in \pi^*(n)\) and \(\tilde{p}(n) \in p^*(n)\). We have \(R(n, p(n)) - R(n, \tilde{p}(n)) \geq 0\), and also \(R(n, p(n)) - R(n, \tilde{p}(n)) = [R(n, p(n)) - AR(n, p(n))] + [AR(n, p(n)) - AR(n, \tilde{p}(n))] + [AR(n, \tilde{p}(n)) - R(n, \tilde{p}(n))]\). The middle term is negative and by [9] the two other terms are \(o(1/n)\) and Part (i) follows.

For Part (ii), we first show that \(p^*\) is a singleton. Suppose that there are two models \(p, q\) in \(p^*\). By the definition of \(p^*\), the components of the projection coefficient vectors \(\beta^{(p)}\) and \(\beta^{(q)}\) must all
be non-zero. Since the function \((Y - a)^2\) is strictly convex in \(a\), we have

\[
\left( Y - \frac{X^{(p)'}\beta^{(p)} + X^{(q)'}\beta^{(q)}}{2} \right)^2 \leq \frac{\left( Y - X^{(p)'}\beta^{(p)} \right)^2 + \left( Y - X^{(q)'}\beta^{(q)} \right)^2}{2},
\]

with equality if and only if \(X^{(p)'}\beta^{(p)} = X^{(q)'}\beta^{(q)}\). Unless \(X^{(p)'}\beta^{(p)} = X^{(q)'}\beta^{(q)}\) a.s., the model \(p \cup q\) would contradict the assumption that \(p\) and \(q\) are in \(\mathcal{M}\) by taking expectations in (39). We assumed that \(E(XX')\) is invertible and hence \(XX'\) vanishes a.s. only for \(\beta = 0\). Adding zeros to \(\beta^{(p)}\) and \(\beta^{(q)}\), thus completing them to vectors in \(\mathbb{R}^d\), we see that the completed vectors are identical. Since the two models are in \(\mathcal{M}\) and their projection coefficients are all non-zero, it follows that \(p = q\), and therefore \(p^*\) is a singleton. The above discussion also shows that any model \(q \in \mathcal{M}\) must satisfy \(q \supseteq p^*\).

In order to show that \(\pi^*(n) = p^*\) for large \(n\), note first that \(q \in \pi^*(n)\) for large enough \(n\) implies \(q \in \mathcal{M}\); if not then there is some \(\tilde{q} \in \mathcal{M}\) such that \(E \left( Y - X^{(\tilde{q})'}\beta^{(\tilde{q})} \right)^2 < E \left( Y - X^{(q)'}\beta^{(q)} \right)^2\). For large \(n\) this \(\tilde{q}\) contradicts \(q \in \pi^*(n)\). It follows that \(q \supseteq p^*\) and it suffices to show that \(tr(\mathbb{V}(p))\) is minimized over \(\mathcal{M}\) by \(p^*\). Indeed we show that if \(p, q \in \mathcal{M}\) and \(q \supseteq p\) then \(tr(\mathbb{V}(p)) < tr(\mathbb{V}(q))\).

Consider a Gram–Schmidt process on the space of square integrable random variables, with the inner product of two random variables being the expectation of their product. Starting with the indexes in \(p\), there exist linear transformations \(\tilde{X}^{(p)} := AX^{(p)}\) and \(\tilde{X}^{(q)} := BX^{(q)}\), where \(A\) and \(B\) are invertible \(p \times p\) and \(q \times q\) matrices, such that \(E \left( \tilde{X}^{(p)}\tilde{X}^{(p)'} \right)\) and \(E \left( \tilde{X}^{(q)}\tilde{X}^{(q)'} \right)\) are both identity matrices (with different dimensions). Also, we can assume that the first \(p\) rows of \(B\) can be obtained from those of \(A\) by adding \(q - p\) zeros to each of these rows. Therefore, for \(k \in p\) we have \(\tilde{X}^{(p)}_k = \tilde{X}^{(q)}_k\), where \(\tilde{X}^{(p)}_k\) is the \(k\)th coordinate of \(\tilde{X}^{(p)}\). The relation \(\tilde{X}^{(p)} = AX^{(p)}\) and straightforward algebra, using properties of the trace function, imply that \(tr(\mathbb{V}(p)) = tr(\mathbb{V}(q))\), where

\[
\mathbb{V}(p) := E \left( \tilde{X}^{(p)}\tilde{X}^{(p)'} \{e^{(p)}\}^2 \right) \left( E \left( \tilde{X}^{(p)}\tilde{X}^{(p)'} \right) \right)^{-1} = E \left( \tilde{X}^{(p)}\tilde{X}^{(p)'} \{e^{(p)}\}^2 \right),
\]

and similarly for \(tr(\mathbb{V}(q))\). We have \(e^{(p)} = e^{(q)}\) for any \(p, q \in \mathcal{M}\) (with probability 1) since otherwise, by the argument in (39), \(p \cup q\) would contradict the assumption that both \(p\) and \(q\) are in \(\mathcal{M}\) as above. We conclude that

\[
tr(\mathbb{V}(p)) = tr \left( E \left( \tilde{X}^{(p)}\tilde{X}^{(p)'} \{e^{(p)}\}^2 \right) \right) = \sum_{k \in p} E \left( \tilde{X}^{(p)}_k e^{(p)} \right)^2 < \sum_{k \in q} E \left( \tilde{X}^{(q)}_k e^{(q)} \right)^2 = tr(\mathbb{V}(q)).
\]

The strict inequality follows from the fact that \(\mathbb{W}\) is positive definite, and thus \(\tilde{X}^{(p)}\tilde{X}^{(p)'} \{e^{(p)}\}^2\) are matrices with positive definite expectations, and therefore positive diagonal elements. Summing up, the above discussion shows that \(p^*\) is a singleton, and that \(tr(\mathbb{V}(p^*))\) is minimal among the models in \(\mathcal{M}\). This implies that \(\pi^*(n) \to p^*\) as \(n \to \infty\). By (39), for every model \(p\), \(R(n, p) - AR(n, p) = o(1/n)\), and therefore \(p^*(n) = \pi^*(n)\) for large enough \(n\). Hence, also \(p^*(n) \to p^*\) as \(n \to \infty\).
Proof of Proposition 2.5 It suffices to prove that $P(\hat{\pi}^*(n, N) = p^*) \to 1$ when $n, N \to \infty$ and $n/N \to 0$, since by Lemma 2.4 $p^*(n) = p^*$ for large enough $n$. Instead, we claim that for every $p \neq p^*$ we have $P(\hat{\pi}^*(n, N) = p) \to 0$, and since there is a finite number of models, the result follows. The latter claim is proved separately for $p \notin \mathcal{M}$ and then for $p \in \mathcal{M}$, (conditions (a) and (b) below):

(a) For $p \notin \mathcal{M}$ we shall show that

$$C(p)(n, N) - C(p^*)(n, N) = A - \frac{tr(V(p)) - tr(V(p^*))}{n} + O_p(1/\sqrt{N}),$$

for a positive constant $A$. Since $\hat{\pi}^*(n, N)$ is the minimizer of $C(p)(n, N)$, it follows that $P(\hat{\pi}^*(n, N) = p) \to 0$ as both $n, N$ go to infinity. To prove (41) note that by the definition of $AR(n, p)$ and Equations (7), (10), and (12), we have

$$C(p)(n, N) - C(p^*)(n, N) = AR(n, p) - AR(n, p^*) + O_p(1/\sqrt{N})$$

$$= E(\frac{1}{n}(Y - X(p) \beta(p))^2 - E(Y - X(p) \beta(p))^2) + \frac{tr(V(p)) - tr(V(p^*))}{n} + O_p(1/\sqrt{N}).$$

Since $p \notin \mathcal{M}$ and $p^* \in \mathcal{M}$, the difference of the expectations, which we denote by $A$, is positive.

(b) For $p \in \mathcal{M}$ and $p \neq p^*$ we shall show that

$$C(p)(n, N) - C(p^*)(n, N) = B/n + O_p(1/\sqrt{N})$$

where $B$ is a positive constant implying that $P(\hat{\pi}^*(n, N) = p) \to 0$ when both $n, N$ go to infinity and $n/N \to 0$.

We now prove (42). Consider $p \in \mathcal{M}$ and $p \neq p^*$. Since both models are in $\mathcal{M}$, we have

$$E(Y - X(p) \beta(p))^2 - E(Y - X(p) \beta(p))^2 = 0$$

and therefore

$$AR(n, p) - AR(n, p^*) = \frac{tr(V(p)) - tr(V(p^*))}{n}.$$ 

In Proposition 2.4 we showed that $tr(V(p^*)) < tr(V(p))$, and therefore $AR(n, p) - AR(n, p^*) = B/n$, where $B$ is a positive constant. Since both $p$ and $p^*$ are in $\mathcal{M}$, it follows that $X(p) \beta(p) = X(p^*) \beta(p^*)$ a.s. (see the proof of Proposition 2.4). Therefore, the first part of $\mathcal{E}_N(p)$ and $\mathcal{E}_N(p^*)$ is equal, and hence, $\mathcal{E}_N(p) - \mathcal{E}_N(p^*) = O_p(1/N)$. Recalling that $AR(n, p) - AR(n, p^*) = B/n$, (10) implies that

$$C(p)(n, N) - C(p^*)(n, N) = B/n + O_p(1/\sqrt{N}) - \frac{tr(V(p)) - tr(V(p^*)) + O_p(1/\sqrt{N})}{n},$$

which implies (12) by (12) (ii).

Proof of Theorem 3.1 The first part follows from (10) of Theorem 2.1. That the $o_p$ terms do not depend on $n$ can be seen by inspecting the proof of (10) of Theorem 2.1. The moreover part follows from the asymptotic normality of each $j$; see (13).
Proof of Proposition 3.3 Part 1 follows from the first part of Theorem 3.1

The proof of Part 2 differs from that of Proposition 2.5 only in taking averages over \( J \) in similar expressions. The only real difference is in case (b), with \( p \) and \( p^* \) both in \( M \). The proof is achieved by showing that there exists \( B > 0 \) such that

\[
\lim_{n,N \to \infty} \sup \ P \left( n \{ C^{(p)}(n,N) - C^{(p^*)}(n,N) \} < B/2 \right) < K/J.
\]

We have

\[
n \{ C^{(p)}(n,N) - C^{(p^*)}(n,N) \} = \frac{1}{J} \sum_{j=1}^J B_j + \frac{1}{J} \sum_{j=1}^J C_j + \frac{1}{J} \sum_{j=1}^J D_j + \frac{1}{J} \sum_{j=1}^J E_j
\]

where,

\[
B_j := \text{tr}(V_j^{(p)}) - \text{tr}(V_j^{(p^*)}), \quad C_j := n(E_{j,N_j}^{(p)} - E_{j,N_j}^{(p^*)}),
\]

\[
D_j := \text{tr}(V_j^{(p)}) - \text{tr}(\tilde{V}_j^{(p)}) - \text{tr}(V_j^{(p^*)}) + \text{tr}(\tilde{V}_j^{(p^*)}),
\]

and \( E_j = n \rho(1/N_j) \), arising from the last term in (10). The proof of (43) is accomplished by showing that \( \frac{1}{J} \sum_{j=1}^J B_j \geq B \), and that the other three sums are small.

Starting with the first term in (44) we consider \( \frac{1}{J} \sum_{j=1}^J B_j = \frac{1}{J} \sum_{j=1}^J \text{tr}(V_j^{(p)}) - \text{tr}(V_j^{(p^*)}) \). Since \( p \) is in \( M \) we have \( p \geq p^* \). By (40) \( \text{tr}(V_j^{(p)}) - \text{tr}(V_j^{(p^*)}) \) is bounded below by \( E_{G_j}(\tilde{X}_k^{(p)}e(p))^2 \)

for \( k \in p \sim p^* \). We have \( \tilde{X}_k = b_k'X \) where \( b_k' \) is the \( k \)th row of the matrix \( B \) defined in the proof of Lemma 2.4 We have \( 1 = E(b_k'X)^2 = b_k'Q_j^{(p)}b_k \) and therefore \( \| b_k'Q_j^{(p)} \| = 1 \). It follows that \( \| b_k \|^2 = b_k'Q_j^{(p)}(1/2)Q_j^{(p)^{-1}}Q_j^{(p)}b_k \geq \lambda_{\text{min}}(\{Q_j^{(p)}\}) = 1/\lambda_{\text{max}}(Q_j^{(p)}) \) and therefore \( E_{G_j}(\tilde{X}_k^{(p)}e(p))^2 = E_{G_j}(b_k'X(p)e)^2 = b_k'W_j^{(p)}b_k \geq \lambda_{\text{min}}(W_j^{(p)})/\lambda_{\text{max}}(Q_j^{(p)}) > 1/C^2 > 0 \).

We obtained that \( \frac{1}{J} \sum_{j=1}^J B_j \geq 1/C^2 =: B \).

We now deal with \( \frac{1}{J} \sum_{j=1}^J C_j \). By (10) and (11) and the fact that \( p, p^* \in M \), this term equals,

\[
\frac{n}{J} \left[ \sum_{j=1}^J \frac{1}{N_j} \left\{ \text{tr}(U_{j,N_j}^{(p)}U_{j,N_j}^{(p^*)}(Q_j^{(p^*)})^{-1}) - \text{tr}(V_j^{(p^*)}) - \text{tr}(U_{j,N_j}^{(p)}U_{j,N_j}^{(p^*)}(Q_j^{(p^*)})^{-1} + \text{tr}(V_j^{(p^*)}) \right\} \right].
\]

Since \( U_{j,N_j}^{(p)} \) converges in distribution to \( Z_j^{(p)} \sim N(0, W_j^{(p)}) \) and \( C \) is an upper bound on \( n/N_j \) we have that the probability that the expression in (45) exceeds \( \varepsilon \) (after taking the limit) is bounded by \( P(T_\sigma > \varepsilon) \) where

\[
T_\sigma := \frac{C}{J} \left[ \sum_j \left\{ \text{tr}(Z_j^{(p)}Z_j^{(p^*)}(Q_j^{(p^*)})^{-1}) - \text{tr}(V_j^{(p^*)}) - \text{tr}(Z_j^{(p)}Z_j^{(p^*)}(Q_j^{(p^*)})^{-1} + \text{tr}(V_j^{(p^*)}) \right\} \right].
\]

Note that the expression within the absolute value sign has mean zero. Writing \( T_\sigma = \frac{C}{J} \sum_{j=1}^J A_j \), Markov’s inequality implies that in order to obtain \( P(T_\sigma > \varepsilon) \leq K/J \) it is enough to bound \( \text{Var}(A_j) \) uniformly in \( j \), which holds when \( (Q_j^{(p)})^{-1}, W_j^{(p)} \) are bounded (element-wise) for all models.
\( p \) (of which there is a finite number) and uniformly for all \( j \). This follows from our eigenvalue assumptions (see (21)) and the fact that the entries of a positive-definite matrix are bounded by its maximal eigenvalue. Finally, it suffices to show that \( D_j \to 0 \) and \( E_j \to 0 \) as \( n, N_j \to \infty \) with \( n/N_j \) bounded. The first follows from (12), and the second is obvious.

**Proof of Lemma 3.5** First notice that when the moments appearing in (i) of Theorem 2.1 are bounded uniformly in \( \theta \), then \( E_{G_\theta}(Y - X' \beta)^2 \) is bounded in \( \theta \). Also, the matrix \( \mathbb{W}_\theta \) is bounded (element-wise) uniformly in \( \theta \). Finally, because \( E\{(X_n'X_n/n)^{-1}\} - Q^{-1}_\theta \) is positive semi-definite (see Groves and Rothenberg [8]), then uniform boundedness of the moment condition (ii) of Theorem 2.1 implies that \( Q^{-1}_\theta \) is uniformly bounded and therefore so is \( tr(V) = tr(\mathbb{W}_\theta Q^{-1}_\theta) \).

We have

\[
AR_{pop}(n, p) - AR(n, p) = \left\{ \int E_{G_\theta}(Y - X' \beta)^2 \mathcal{P}(d\theta) - \frac{1}{J} \sum_{j=1}^J E_{G_j}(Y - X' \beta_j)^2 \right\} + \frac{1}{n} \left\{ \int tr(V) \mathcal{P}(d\theta) - \frac{1}{J} \sum_{j=1}^J tr(V_j) \right\}.
\]

(46)

The above two sums contain random variables that are bounded, and hence so are their variances. The central limit theorem applied twice, implies (24) and the claimed asymptotic normality. It is easy to see directly from (46) that the \( \Omega_p \) term in (24) is uniform in \( p \).

For the proof of Proposition 3.6 we need the following lemma:

**Lemma 7.1.** Suppose that the conditions of Lemma 3.4 hold and also that \( \lambda_{\min}(\mathbb{W}_\theta) \) is bounded away from zero uniformly in \( \theta \); then

1. The set \( p_{\text{pop}}^* \) is a singleton and as \( n \to \infty \) both \( \pi_{\text{pop}}^*(n) \to p_{\text{pop}}^* \) and \( p_{\text{pop}}^*(n) \to p_{\text{pop}}^* \), and therefore also \( \pi_{\text{pop}}^*(n) = p_{\text{pop}}^*(n) \) for large \( n \).

2. There exists a constant \( K_C \) depending only on \( C \), such that for \( \pi^*(n) \) defined in \( (20) \),

\[
P(\pi^*(n) \leq \pi_{\text{pop}}^*(n)) \geq 1 - \frac{K_C}{J}.
\]

**Proof of Lemma 7.1.** Part 1. The proof is similar to that of Proposition 2.4. We sketch the proof. Let \( p \) and \( q \) be in \( p_{\text{pop}}^* \). By convexity as in (39),

\[
\frac{(Y - X^{(p)' \beta^{(p)})^2} + (Y - X^{(q)' \beta^{(q)})^2)}{2} - \left( Y - \frac{X^{(p)' \beta^{(p)} + X^{(q)' \beta^{(q)}}}{2} \right)^2 \geq 0,
\]

(47)

with equality iff \( X^{(p)' \beta^{(p)} = X^{(q)' \beta^{(p)}} \). Since \( p \) and \( q \) are in \( M_{\text{pop}} \), the expectation of the left-hand side of (47) is zero. It follows that \( \int P_{G_\theta}(X^{(p)' \beta^{(p)}) = X^{(q)' \beta^{(q)}}) \mathcal{P}(d\theta) = 1 \), and therefore for every model \( p \) in \( M_{\text{pop}} \) we have that \( p_{\text{pop}}^* \subseteq p \). By the assumptions on moments being uniformly bounded, it follows that \( \lambda_{\max}(Q_\theta) \) is bounded above and \( \lambda_{\min}(\mathbb{W}_\theta) \) is positive and bounded away
from zero, both uniformly in $\theta$. Now (40) and the discussion in the paragraph above (45) imply that if $p^\ast_{\text{pop}} \subseteq p \in \mathcal{M}_{\text{pop}}$ and $p^\ast_{\text{pop}} \neq p$ then $\int \text{tr} \left( \psi_\theta^{|p^\ast_{\text{pop}}|} \right) \mathcal{P}(d\theta) < \int \text{tr} \left( \psi_\theta^{|p^\ast|} \right) \mathcal{P}(d\theta)$. Therefore, $p^\ast_{\text{pop}}$ has a minimal trace among $\mathcal{M}_{\text{pop}}$. It follows that $\pi^\ast_{\text{pop}}(n) \to p^\ast_{\text{pop}}$ as $n \to \infty$.

Furthermore, Lemma 3.4 implies that $\pi^\ast_{\text{pop}}(n)$ and $p^\ast_{\text{pop}}(n)$ coincide for large $n$. The result now follows from the convergence of $\pi^\ast_{\text{pop}}(n)$ to $p^\ast_{\text{pop}}$.

**Part 2.** Recall (46). Let $-\delta$ be the difference between the minimum of $\text{AR}_{\text{pop}}(n,p)$, attained at $\pi^\ast_{\text{pop}}(n)$, and the smallest next value. Since $\text{Var}(E_{\mathcal{G}_\theta}(Y - X' \beta_\theta)^2)$ and $\text{Var}(\text{tr}(V_\theta))$ are finite, we have by Chebyshev’s inequality and the union bound that

$$P(\exists p | \text{AR}_{\text{pop}}(n,p) - \text{AR}(n,p)| > \varepsilon) \leq \frac{C}{\varepsilon^2 J},$$

for any $\varepsilon > 0$, and in particular for $\varepsilon = \delta/2$, where $C$ does not depend on $n$ and $J$. If the event $\{\forall p | \text{AR}_{\text{pop}}(n,p) - \text{AR}(n,p)| \leq \varepsilon\}$ occurs for all $p$ then any minimizer of $\text{AR}(n,p)$ also minimizes $\text{AR}_{\text{pop}}(n,p)$.

**Proof of Proposition 3.6** The first part of Proposition 3.6 follows from Part 1 of Proposition 3.3, which shows that $\hat{\pi}^\ast(n,N) \subseteq \pi^\ast(n)$ with probability converging to 1, and Part 2 of Lemma 7.1 which shows that $\pi^\ast(n) \subseteq \pi^\ast_{\text{pop}}(n)$ with high probability.

The second part of Proposition 3.6 follows from a combination of several statements: $\hat{\pi}^\ast(n,N) = p^\ast(n)$ with high probability (Proposition 3.3 Part 2); $p^\ast(n) = \pi^\ast(n)$ for large $n$ (Proposition 3.2); $\pi^\ast(n) \subseteq \pi^\ast_{\text{pop}}(n)$ with high probability (Lemma 7.1 Part 2); and for large $n$ $\pi^\ast_{\text{pop}}(n)$ is a singleton, and $\pi^\ast_{\text{pop}}(n) = p^\ast_{\text{pop}}(n)$ (Lemma 7.1 Part 1).
## Appendix B: A table of notation

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J )</td>
<td>Number of observed regression datasets</td>
</tr>
<tr>
<td>( N_j )</td>
<td>Number of observations in the the ( j )th regression dataset</td>
</tr>
<tr>
<td>( Y_{ij} )</td>
<td>The response of the ( i )th observation from the ( j )th regression</td>
</tr>
<tr>
<td>( X_{ij} \in \mathbb{R}^d )</td>
<td>The covariate vector of the ( i )th observation from the ( j )th regression</td>
</tr>
<tr>
<td>( (X, Y) )</td>
<td>A generic observation (whose distribution is ( G_j ))</td>
</tr>
<tr>
<td>( D_j = {(X_{ij}, Y_{ij})} )</td>
<td>The ( j )th regression dataset</td>
</tr>
<tr>
<td>( G_j )</td>
<td>The distribution of the ( j )th regression, i.e., ( {(X_{ij}, Y_{ij})} \sim iid G_j )</td>
</tr>
<tr>
<td>( G )</td>
<td>A set of distributions to which ( G_j ) belongs (the cases (</td>
</tr>
<tr>
<td>( p )</td>
<td>A model; subset of ( {1, \ldots, d} ) (denotes also the number of covariates)</td>
</tr>
<tr>
<td>( R(n,p) )</td>
<td>The prediction error of model ( p ) with ( n ) observations for the case (</td>
</tr>
<tr>
<td>( AR(n,p) )</td>
<td>Approximate prediction error; ( AR(n,p) ) and ( AR_{pop}(n,p) ) are approximations of ( R(n,p) ) and ( R_{pop}(n,p) ), respectively</td>
</tr>
</tbody>
</table>

In the notation below \( j \) and \( (p) \) are sometimes suppressed

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{j,N_j}^{(p)} )</td>
<td>The ( N_j \times p ) design matrix of the ( j )th regression</td>
</tr>
<tr>
<td>( Y_{j,N_j} )</td>
<td>The vector of responses for the ( j )th regression</td>
</tr>
<tr>
<td>( \beta_{j}^{(p)} )</td>
<td>Projection coefficients under ( G_j ) for model ( p )</td>
</tr>
<tr>
<td>( e_{j}^{(p)} )</td>
<td>The residual; ( e_{j}^{(p)} = Y - X_{j}^{(p)} \beta_{j}^{(p)} ); ( e_{j,N_j} ) denotes the vector of the residuals of dimension ( N_j )</td>
</tr>
<tr>
<td>( \hat{\beta}_{j,n}^{(p)} )</td>
<td>The least squares estimate of ( \beta_{j}^{(p)} ) based on ( n ) observations.</td>
</tr>
<tr>
<td>( Q_{j}^{(p)} )</td>
<td>( Q_{j}^{(p)} = E_{G_j}(X_{j}^{(p)}X_{j}^{(p)'} \sigma^2) )</td>
</tr>
<tr>
<td>( Q_{j}^{(p)} )</td>
<td>( Q_{j}^{(p)} ) denotes the vector of the residuals of dimension ( N_j )</td>
</tr>
<tr>
<td>( \hat{Q}_{j,N_j}^{(p)} )</td>
<td>The empirical estimate of ( Q_{j}^{(p)} )</td>
</tr>
<tr>
<td>( \hat{\Psi}_{j,N_j}^{(p)} )</td>
<td>The empirical estimate of ( \hat{Q}_{j,N_j}^{(p)} )</td>
</tr>
<tr>
<td>( U_{j,N_j}^{(p)} )</td>
<td>( U_{j,N_j}^{(p)} = \frac{1}{\sqrt{N_j}}X_{j,N_j}^{(p)}e_{j,N_j} ) (it is not a statistic)</td>
</tr>
</tbody>
</table>

\( C^{(p)}(n,N) \) An estimate of \( AR(n,p) \); \( C^{(p)}(n,N) \) corresponds to the case \( J > 1 \); \( C^{(p)}(n,N) \) and \( C^{(p)}(n,N) \) denote a jackknife bias correction

\( p^*(n) \) \( arg \min_p R(n,p) \) (the best model for \( n \) observations)

\( \pi^*(n) \) \( arg \min_p AR(n,p) \)

\( p^* \) The limit of both \( p^*(n) \) and \( \pi^*(n) \) as \( n \to \infty \)

\( \hat{\pi}^*(n,N) \) \( arg \min_p C^{(p)}(n,N) \)