Optimal designs for the development of personalized treatment rules

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Abstract

In the present paper, personalized treatment means choosing the best treatment for a patient while taking into account certain relevant personal covariate values. We study the design of trials whose goal is to find the best treatment for a given patient with given covariates. We assume that the subjects in the trial represent a random sample from the population, and consider the allocation, possibly with randomization, of these subjects to the different treatment groups in a way that depends on their covariates.

We derive approximately optimal allocations, aiming to minimize expected regret, assuming that future patients will arrive from the same population as the trial subjects. We find that, for the case of two treatments, an approximately optimal allocation design does not depend on the value of the covariates but only on the variances of the responses. In contrast, for the case of three treatments the optimal allocation design does depend on the covariates as we show for specific scenarios. Another finding is that the optimal allocation can vary a lot as a function of the sample size, and that randomized allocations are relevant for relatively small samples, and may not be needed for very large studies.

Key words: personalized medicine; minimal regret; optimal allocation; experimental design.

1 Introduction

This work concerns the study of optimal designs of clinical trials with the goal of estimating personalized treatment rules. We consider multi-armed clinical trials for treatments, where the primary outcome may depend on patients’ covariates. Modeling the outcome a function of the covariates does not only reduce
the error variance, but allows one also to estimate “personalized” treatment rules. These rules assign the treatment with the best estimated outcome for a patient having given values of the covariates.

In the related context of bandit problems with covariates, Goldenshluger and Zeevi (2013) studied a sequential allocation scheme where at each stage one out of two treatments needs to be assigned under a minimax framework. A high-dimensional version of this problem was studied in Bastani and Bayati (2015). The goal in the Bandit formulation is to determine the best treatment while minimizing some loss or regret function for subjects in the trial sample. This is a different setting than the one considered here, since rather than be concerned with optimal treatment for subjects in the trial sample, we look for designs that will allocate patients to treatments efficiently for the goal of finding the best treatment for future patients who may require one of the treatments under study. Personalized design as a function of covariates appears in many papers, such as Qian and Murphy (2011), Tian et al. (2014), Ballarini et al. (2018), or Pan and Zhao (2020). However, these papers are concerned with inference about the optimal treatment rather than the design of the trial, which is the goal of this paper.

**Problem statement:** Consider $K$ possible treatments $T_1, \ldots, T_K$. Let $Y$ be a one-dimensional continuous response variable and let $X \in \mathbb{R}^p$ be a vector of a subject’s covariates. We assume that the joint distribution of the covariates is continuous with density denoted by $f(x)$. The expected response is $E(Y|X, T_k) = g_k(X)$, where $g_k$ is an unknown function. The optimal treatment for a subject with covariate $x$ is $\delta^*(x) = \arg\max_{k \in \{1, \ldots, K\}} g_k(x)$ (assuming that a higher response is better). If the above arg max contains more than one $k$, one arbitrary treatment is selected.

Suppose that a clinical trial with $n$ subjects is performed in order to estimate $\delta^*$. Let $X_1, \ldots, X_n$ denote a sample according to $f$ of the covariates of the $n$ subjects. The design we study here consists of allocating subjects to treatments, taking account of their covariates. Each subject $i$ is allocated independently to treatment $k$ with probability $\pi_k(X_i)$, where $\pi_1(x), \ldots, \pi_K(x)$ are non-negative functions satisfying for each $x$, $\sum_{k=1}^K \pi_k(x) = 1$. The allocation functions define densities for the covariates in the $K$ treatment groups. Specifically, in treatment $k$, $X$ is sampled from the density $f_k(x) := f(x)\pi_k(x)/\nu_k$ where $\nu_k := \int f(x)\pi_k(x)dx$. Our purpose is to find the optimal allocation functions $\pi_1(x), \ldots, \pi_K(x)$ that minimize the regret, which is defined next.

Let $\hat{\delta}$ denote the final estimate of the optimal rule $\delta^*$ based on the covariates, responses, and allocations of the subjects in the trial. Then the regret for a randomly chosen future patient is defined as

$$R(\pi_1, \ldots, \pi_K) := E\left[g_{\delta^*}(\tilde{X}) - g_{\hat{\delta}}(\tilde{X})\right],$$

where $\tilde{X}$ denotes the covariate vector of the chosen patient and the expectation is over $\tilde{X}$, $\hat{\delta}$, and the randomizations in the allocations. In words, the regret is the difference between the expected response
to the optimal treatment $\delta^*$ and its estimate $\hat{\delta}$ for any independent future subject arising from the same population as those in the trial. A similar criterion appears in Qian and Murphy (2011). Our goal is to minimize the regret over all feasible allocations, that is, non-negative functions $\pi_k(\cdot)$ that satisfy $\sum_{k=1}^K \pi_k(x) = 1$ for every $x$.

In this study we assume that we have some preliminary knowledge about the parameters of the problem, such as the functions $g_k$, and the variances of $Y$ given $X, T_k$ and the density $f$ of the covariates. This approach concurs with existing literature on locally optimal designs; see, e.g., Chernoff (1953) and Silvey (2013, Chapter 6), where optimality is achieved for a given set of parameters. Our goal is to find optimal allocation functions relative to the assumed parameter values, and obtain a treatment rule $\hat{\delta}$ based on an experiment according to our design. Knowledge about the parameters can arise from previous experience or theory, from a pilot study, or in some situations from earlier phases of the study. Other approaches, like sequential or Bayesian designs will not be discussed here. However, our approach suggests a natural adaptive formulation where after observing an initial part of the sample one estimates the parameters and aims at an optimal design relative to these estimates, and the process can be repeated.

A trial of the kind we study ends with recommendations on treatments, amounting to a claim of causality. Here we allow the allocation to treatments to be a function of the covariate vector $X$ only, and thus conditioned on $X$, the allocation and the response are obviously independent. In this paper we shall assume that a linear model holds true (see Section 2); however, assuming any tractable parametric model on $E(Y|X)$ can be handled in the same way, using non-linear least squares. Assuming a model, linear or more complex, rather than taking the approach that “all models are wrong”, is necessary if one wants to avoid a separate trial for each level of $X$, which seems inefficient. A linear model assumption may be reasonable if the range of $X$ is limited, which is often very natural. Another possibility is to consider a polynomial linear model of some degree rather than a simple linear model (see Section 3.3). The parameters of any model for $E(Y|X)$ obviously depend only on the conditional distribution of $Y$ given $X$, allowing us to use them for causal inference given $X$; see, e.g., Hernán and Robins (2010). This remains true even if the allocation functions are “deterministic”, taking only the values 0 and 1. On the other hand, suppose we use the best linear predictors without assuming a linear model. In this case, the parameters of the best linear predictor (projection coefficients) depend on the distribution of $X$. In this case, causality cannot be claimed. Starting in Section 2 we shall assume a linear model.

**Main results:** We study the regret via an approximation that assumes “ideal” conditions, such as normality instead of asymptotic normality of linear regression coefficient estimators. We show that this yields a useful approximation for quite small sample sizes. We also study an asymptotic regret. It turns out that when there are three or more treatments, optimal designs for moderate sample sizes may be
totally different from the asymptotic design, and in this sense one should beware of asymptotics.

For the case of two treatments, we show that the optimal trial design allocates patients in proportion to the standard deviations of the response under each treatment, and the optimal allocation probabilities do not depend on the covariates. This is shown for the case where the outcome in each treatment group depends on the covariates according to a \( p \)-dimensional linear regression model, as well as for the case of a single covariate and a polynomial regression model.

For the case of three or more treatments and a linear regression model in a single covariate, we combine theoretical arguments and numerical calculations to study some examples of optimal designs. This includes a study comparing three treatments (see Ebbeling et al. (2018)) based on data that are reconstructed from a published paper; we compare the design in this paper to an optimal design obtained by our approach. We show that the optimal allocation in general depends on the covariate. For specific cases, we show that the asymptotically optimal allocation is deterministic, i.e., the range of the covariate is partitioned into intervals and in each interval the allocation probability to a particular treatment is 1. Such allocation rules arise typically in very large studies.

2 The regret under “ideal” conditions

Let \((Y_1, X_1), \ldots, (Y_n, X_n)\) be a sample obtained by given allocation functions \(\pi_1, \ldots, \pi_n\), along with the treatment allocated for each \(X_i\). Assume \(K \geq 2\) and \(X \in \mathbb{R}^p\), and under treatment \(T_k\) we have \(Y = g_k(X) + \varepsilon\), where \(g_k(X) = E(Y|X, T_k) = \alpha_k + \beta_k^tX\), and \(\sigma_k^2 := Var(\varepsilon|X, T_k)\). Let \(\hat{\alpha}_k, \hat{\beta}_k\) denote the OLS estimators based on data from treatment \(T_k\), \(\hat{g}_k(X)\) the corresponding estimated regression functions, and \(\hat{\delta}(x) = \arg\max_k \hat{g}_k(X)\) the decision rule. Then,

\[
R(\pi_1, \ldots, \pi_K) = \sum_{k=1}^K \int_{\mathbb{R}^p} P(\hat{\delta}(x) = k) \left[ g_{\delta^*(x)}(x) - g_k(x) \right] f(x) dx. \tag{2}
\]

For \(k = 1, \ldots, K\), let

\[
Q_k := \int_{\mathbb{R}^p} \left( \frac{1}{x} \right)(1, x^t)f_k(x)dx, \quad \Sigma_k := \frac{1}{\nu_k^2}Q_k^{-1}, \quad \xi_k^2(x) := (1, x^t)\Sigma_k\left( \frac{1}{x} \right). \tag{3}
\]

It is well known that \(\sqrt{n}[(\hat{\alpha}_k, \hat{\beta}_k) - (\alpha_k, \beta_k)] \to N(0, \Sigma_k)\), and in particular \(\Sigma_k\) is the asymptotic variance of the OLS estimators.

In order to approximate the probability \(P(\hat{\delta}(x) = k)\), we first impose “ideal” conditions, to be dropped later: 1. The asymptotic variance \(\Sigma_k\) is the exact variance matrix of the OLS estimators and thus \(\xi_k^2(x)/n = Var(\hat{g}_k(x))\); and 2. The error \(\varepsilon|X, T_k \sim N(0, \sigma_k^2)\). Under the ideal conditions the
estimators $\hat{g}_k(x)$ are jointly normal, and independent conditionally on $X_1, \ldots, X_n, T_1, \ldots, T_K$, with constant expectation $\alpha_k + \beta'_k x$, and by normality they are independent. Therefore, with $P_I$ and $E_I$ denoting the probability and expectation under the ideal conditions, we have

$$P_I \left( \hat{\delta}(x) = k \right) = P_I \left( \max_{l = 1, \ldots, K, l \neq k} \hat{g}_l(x) < \hat{g}_k(x) \right)$$

$$= E_I \prod_{l \neq k} P_I \left( \frac{\sqrt{n} (\hat{g}_l(x) - g_l(x))}{\xi_l(x)} < \frac{\sqrt{n} (\hat{g}_k(x) - g_l(x))}{\xi_k(x)} \mid \hat{g}_k(x) \right)$$

$$= E_I \prod_{l \neq k} P \left( Z < \frac{\sqrt{n} (\hat{g}_k(x) - g_l(x))}{\xi_k(x)} \mid \hat{g}_k(x) \right) = E \prod_{l \neq k} P \left( \tilde{Z} < \frac{Z \xi_l(x) + \sqrt{n} (g_k(x) - g_l(x))}{\xi_k(x)} \mid Z \right)$$

$$= \int_{t=1, \ldots, K, l \neq k} \Phi \left( \frac{z \xi_l(x) + \sqrt{n} (g_k(x) - g_l(x))}{\xi_k(x)} \right) \varphi(z) dz,$$

where the $E$ and $P$ in the penultimate expression are with respect to $Z$ and $\tilde{Z}$, which are independent $N(0, 1)$, and $\tilde{Z}$ represents $\sqrt{n} \hat{g}_k(x) - g_l(x)) \xi_k(x)$; $\varphi$ and $\Phi$ denote the standard normal density and cumulative distribution function. The product in the second line was justified above by independence under the ideal conditions, which are used also in both equalities of the third line. We define the ideal regret, denoted by $R_I$, by

$$R_I(\pi_1, \ldots, \pi_K) := \sum_{k=1}^K \int_{\mathbb{R}^p} P_I \left( \hat{\delta}(x) = k \right) \left[ g_{\pi_k}(x) - g_k(x) \right] f(x) dx,$$

where $P_I \left( \hat{\delta}(x) = k \right)$ is given in (4).

3 The case of two treatments

3.1 Regret approximations and optimal design

We assume that $K = 2$ and $Y = g_k(X) + \varepsilon$ under treatment $T_k$ for $k = 1, 2$, where $g_k(X)$ denotes the conditional mean of $Y$ given $X, T_k$, $X \in \mathbb{R}^p$ and

$$g_1(X) = \alpha_1 + \beta'_1 X, \ g_2(X) = \alpha_2 + \beta'_2 X, \ Var(\varepsilon \mid X, T_1) = \sigma_1^2 \text{ and } Var(\varepsilon \mid X, T_2) = \sigma_2^2.$$

We do not assume normality of the errors. However, we assume the existence of the moment generating function of $\varepsilon$ when conditioned on $X, T_k$. This assumption is needed for some large deviation exponential bounds used below, and could be relaxed to assuming finiteness of some high order moments instead. We also assume that $X$ is continuous with a bounded density $f(x)$ supported on $[0, 1]^p$. 
Under the ideal conditions of Section 2 with $K = 2$, Equation (4) becomes

$$P_I(\delta(x) = k) = \Phi \left( \frac{\sqrt{n}|g_k(x) - g_l(x)|}{\sqrt{V(x)}} \right),$$

where $V(x) := \xi_1^2(x) + \xi_2^2(x) = (1, x^T) (\Sigma_1 + \Sigma_2) \left( \frac{1}{x} \right)$ (7)

for $1 \leq k \neq l \leq 2$. The ideal regret (5) is

$$R_I(\pi_1, \pi_2) = \int_{\{x: g_1(x) > g_2(x)\}} \phi \left( \frac{\sqrt{n}|g_2(x) - g_1(x)|}{\sqrt{V(x)}} \right) [g_1(x) - g_2(x)] f(x) dx + \int_{\{x: g_1(x) < g_2(x)\}} \phi \left( \frac{\sqrt{n}|g_1(x) - g_2(x)|}{\sqrt{V(x)}} \right) [g_2(x) - g_1(x)] f(x) dx$$

$$= \int_{[0,1]^p} \phi \left( -\frac{\sqrt{n}|g_1(x) - g_2(x)|}{\sqrt{V(x)}} \right) [g_2(x) - g_1(x)] f(x) dx. \quad (8)$$

By (8), minimization of $R_I(\pi_1, \pi_2)$ amounts to finding the allocation functions $\pi_1, \pi_2$ that minimize $V(x)$. The following theorem states that this can be done by minimizing $V(x)$ uniformly over $x$.

**Theorem 3.1.** The allocation functions $\pi_1(x) := \frac{\sigma_1}{\sigma_1 + \sigma_2}$ and $\pi_2(x) := \frac{\sigma_2}{\sigma_1 + \sigma_2}$ minimize both $V(x)$ uniformly over $x$, and $R_I(\pi_1, \pi_2)$.

Recalling that $V(x)$ can be written as a quadratic form, see (7), Theorem 3.1 follows immediately from Lemma 3.2 below whose proof, as all other proofs, is given in the Appendix. Given matrices $A$ and $B$, we write $A \succeq B$ if $A - B$ is positive semi-definite. Let $Q := \int_{[0,1]^p} \left( \frac{1}{x} \right) (1, x) f(x) dx$, and recall the notation of (3). We have

**Lemma 3.2.** With the above definitions,

$$\Sigma_1 + \Sigma_2 \succeq (\sigma_1 + \sigma_2)^2 Q^{-1}.$$ \hspace{1cm} (9)

Moreover, the lower bound is attained when $\pi_1(x) = \pi_1^0(x)$ and $\pi_2(x) = \pi_2^0(x)$.

To see the latter statement note that when $\pi_k(x)$ do not depend on $x$, both matrices $Q_k$ are proportional to $Q$ and therefore both matrices $\Sigma_k$ are proportional to $Q^{-1}$.

We now discuss the regret (2). In Theorem 3.3 we approximate the regret by the ideal regret, and then we use the above results for the ideal regret to find an asymptotically optimal design for the regret itself. We later demonstrate numerically that the latter design provides a good approximation for finite
sample sizes. See also Theorem 3.3. The regret is given by

\[ R(\pi_1, \pi_2) = \int_{\{x: g_1(x) > g_2(x)\}} P(\tilde{g}_2(x) > \tilde{g}_1(x))(g_1(x) - g_2(x))f(x)dx + \int_{\{x: g_2(x) > g_1(x)\}} P(\tilde{g}_1(x) > \tilde{g}_2(x))(g_2(x) - g_1(x))f(x)dx. \] (10)

**Theorem 3.3.** Under model (6) with \( \nu_1, \nu_2 > 0 \), we have for any \( \varepsilon > 0 \)

\[ n^{3/2-\varepsilon} |R(\pi_1, \pi_2) - R_I(\pi_1, \pi_2)| \to 0 \text{ as } n \to \infty. \]

To facilitate the representation, we state the next result in the one-dimensional case first. In this case \( g_1 \) and \( g_2 \) are functions of a single variable, and we assume that their intersection point \( \theta := \frac{\alpha_1}{\beta_2 - \beta_1} \) is in \([0, 1]\). Assume without loss of generality, that \( \beta_2 \geq \beta_1 \). It follows that for \( x < \theta, g_1(x) > g_2(x) \), that is, \( T_1 \) is the better treatment, and when \( x > \theta, T_2 \) is better. In the case \( p = 1 \) the limit of the regret is simple, given by the following theorem.

**Theorem 3.4.** Under model (6) with \( \nu_1, \nu_2 > 0 \), then

\[ \lim_{n \to \infty} nR(\pi_1, \pi_2) = \frac{V(\theta) f(\theta)}{2(\beta_2 - \beta_1)}. \]

It follows that in order to minimize \( \lim_{n \to \infty} nR(\pi_1, \pi_2) \) we have to minimize \( V(\theta) \), which by Theorem 3.1 is achieved by taking \( \pi_1(x) = \pi_1^0(x) \) and \( \pi_2(x) = \pi_2^0(x) \).

We remark that by a delta method calculation, the asymptotic variance of \( \hat{\theta} = \frac{\hat{\alpha}_1 - \hat{\alpha}_2}{\hat{\beta}_2 - \hat{\beta}_1} \) is \( \frac{V(\theta)}{(\beta_2 - \beta_1)^2} \), which is proportional to the limit of \( nR(\pi_1, \pi_2) \). Thus, minimizing this limit is equivalent to minimizing the asymptotic variance of the estimator of the intersection point \( \theta \).

We return to general dimension \( p \). Set \( \beta_k := (\beta_{k,1}, \ldots, \beta_{k,p}) \), \( k = 1, 2 \), \( \beta_{k,-1} := (\beta_{k,2}, \ldots, \beta_{k,p}) \) and without loss of generality let \( \beta_{2,1} \geq \beta_{1,1} \). For given \( x_{-1} = (x_2, \ldots, x_p) \) define \( \theta_1 := \theta_1(x_{-1}) := \frac{\alpha_1 - \alpha_2 + (\beta_{1,-1} - \beta_{2,-1})' x_{-1}}{\beta_{2,1} - \beta_{1,1}} \). If \( \theta_1 \in [0, 1] \), treatment \( T_1 \) is better for a covariate vector \( x \) satisfying \( x_1 < \theta_1 = \theta_1(x_{-1}) \), and otherwise \( T_2 \) is better. The limit of the regret is given in the following theorem.

**Theorem 3.5.** Assume model (6) and \( \nu_1, \nu_2 > 0 \), then

\[ \lim_{n \to \infty} nR(\pi_1, \pi_2) = \frac{1}{2(\beta_{2,1} - \beta_{1,1})} \int_{[0,1]^{p-1}} V(\theta_1, x_{-1}) f(\theta_1, x_{-1}) I\{\theta_1 \in [0,1]\} dx_{-1}. \] (11)

Thus, asymptotically, \( nR(\pi_1, \pi_2) \) is proportional to the integral of the variance of the estimate of the intersection curve \( \theta_1(x_{-1}) \), with weights proportional to the density at the intersection points and by (11) \( \lim_{n \to \infty} nR(\pi_1, \pi_2) \) is minimized by the allocation \( \pi_1^0(x), \pi_2^0(x) \) since this allocation minimizes \( V \) uniformly.

The following corollary of Theorem 3.3 shows that \( \pi_1^0, \pi_2^0 \), which minimize the ideal, regret are also approximately optimal for the regret \( R(\pi_1, \pi_2) \) itself.
Corollary 3.1. For any \( \varepsilon > 0 \) there exists \( C > 0 \) such that for any allocation functions \( \pi_1, \pi_2 \) with \( \nu_1, \nu_2 > 0 \)

\[
R(\pi_1, \pi_2) \geq R(\pi_0^1, \pi_0^2) - \frac{C}{n^{3/2-\varepsilon}}.
\]

In particular, the optimal design that minimizes \( R(\pi_1, \pi_2) \) can improve the allocation \( \pi_0^1, \pi_0^2 \) by at most \( C/n^{3/2-\varepsilon} \).

The error term is meaningful as it is of smaller order than the regret, which is \( 1/n \) according to Theorem 3.5.

3.2 A numerical example

To illustrate the asymptotic approximations of the regret and the optimal allocations derived in Section 3.1 consider an example of model (6) with \( p = 1, \alpha_1 = 0.2, \alpha_2 = 0, \beta_1 = 0.5, \beta_2 = 1, \sigma_1^2 = 0.1, \sigma_2^2 = 0.2, \) and \( X \sim U[0,1] \). The optimal allocation is \( \nu_1 = 0.414 \). The \( \sigma \)'s are chosen such that \( R^2 \) in both regressions is about 0.2. Figure 1(a) shows the regression lines. For \( x \) smaller than \( \theta = 0.4 \), treatment 1 is better and otherwise treatment 2 is preferred. The regret (10) is evaluated using simulations (with \( 10^5 \) replications) and is compared to the ideal regret (8). Two scenarios are considered for the residuals in the regression model: normal and centered exponential. Figures (b) and (c) show the regret and ideal regret for \( n = 100 \) under the allocation function \( \pi_1(x) = \nu_1 \) where \( \nu_1 \) vary from 0.2 to 0.6. The allocation optimizing the ideal regret is marked with a vertical line. While there is a slight deviation between the ideal and actual regret, it seems that the optimal allocations, with respect to both, are very close. Thus, the approximation of Theorem 3.3 works well also for small \( n \) in this example. Figure 1 (d) shows \( n R_I(\pi_1, \pi_2) \) of the balanced design \( (\pi_1 = \pi_2 = 1/2) \) for large \( n \) and its limit, see Theorem 3.4. It is shown that the regret slowly converges to its limit, and it is close to the limit only when \( n \) is quite large.
Figure 1: Figure (a): the regression lines. Figures (b) and (c) show the regret (computed by a simulation with 95% confidence intervals), and the ideal regret for \( n = 100 \) and different allocation ratios, where in (b) the residual is normal and in (c) it is exponential (centered). The regret is calculated for the allocation function \( \pi_1(x) = \nu_1 \) where \( \nu_1 \) varies from 0.2 to 0.6. The optimal allocation is marked by the vertical line. Figure (d) shows \( n \) times the ideal regret (blue line) and its limit (gray line).
3.3 Polynomial Regression

We now consider the model of (6) where the functions $g_k(x)$ are assumed to be polynomials of a single continuous covariate $X$ with density $f$, i.e., $g_k(X) = \alpha_k + \sum_{j=1}^J \beta_{jk}X^j$, where $J$ denotes the degree of the polynomial. Let $\theta_1 < \cdots < \theta_L$, where $L \leq J$, be the crossing points of $g_1(x)$ and $g_2(x)$, and define $\theta_0 = 0$ and $\theta_{L+1} = 1$. Assume without loss of generality that $g_1(x) > g_2(x)$ for $x \in (\theta_\ell, \theta_{\ell+1})$ if $\ell$ is odd, and the reverse inequality holds when $\ell$ is even. Therefore, the regret is

$$R(\pi_1, \pi_2) = \sum_{\ell \text{ is odd}} \int_{\theta_\ell}^{\theta_{\ell+1}} P(\hat{g}_2(x) > \hat{g}_1(x))(g_1(x) - g_2(x))f(x)dx$$

$$+ \sum_{\ell \text{ is even}} \int_{\theta_\ell}^{\theta_{\ell+1}} P(\hat{g}_1(x) > \hat{g}_2(x))(g_2(x) - g_1(x))f(x)dx.$$

Notice that Theorems 3.3 and 3.5, which provide results for multivariate $X$, do not apply in this case because the random vector $X = (X, X^2, \ldots, X^J)$ does not have a joint density. However, using the arguments of Theorem 3.3 (under similar notation and regularity conditions) it can be shown that $n^{3/2-\varepsilon}|R(\pi_1, \pi_2) - R_I(\pi_1, \pi_2)| \to 0$, for any $\varepsilon > 0$, where

$$R_I(\pi_1, \pi_2) = \int_0^1 \Phi \left( \frac{-\sqrt{n}|g_2(x) - g_1(x)|}{\sqrt{V(x)}} \right) |g_2(x) - g_1(x)|f(x)dx,$$

(12)

and now $x^t = (1, x, \ldots, x^J)$, $Q_k = \int_{\mathbb{R}^p} xx^tf_k(x)dx$ and $\Sigma_k = \frac{1}{\nu_k} \sigma_k^2 Q_k^{-1}$, $k = 1, 2$, and $V(x) = x^t(\Sigma_1 + \Sigma_2)x$, which is the asymptotic variance of $\hat{g}_1(x) - \hat{g}_2(x)$.

Lemma 3.2 implies that the design where $\pi_1(x) = \frac{\sigma_1}{\sigma_1+\sigma_2}$ minimizes $V(x)$ uniformly over all $x$’s. It follows that this design is asymptotically optimal also for this problem.

By Proposition A.1 (in the appendix) we have

$$\lim_{n \to \infty} nR(\pi_1, \pi_2) = \sum_{\ell=1}^L \frac{f(\theta_\ell)V(\theta_\ell)}{2(|\beta_2 - \beta_1|)^t} \zeta_\ell,$$

where $\zeta_\ell := (1, 2\theta_\ell, \ldots, J\theta_\ell^{J-1})^t$.

A careful inspection of the proofs of the above results shows that the optimality of the design with $\pi_1(x) = \frac{\sigma_1}{\sigma_1+\sigma_2}$ generalizes to regression functions of the form $g_k(X) = \sum_j \beta_{jk}h_j(X)$ for any functions $h_j$ having bounded derivatives.
4 \; K \; Treatments \; and \; one \; Covariate

4.1 \; Regret \; and \; ideal \; regret

We consider the case \( p = 1 \) and \( g_k(X) = \alpha_k + \beta_k X \), \( \text{Var}(Y|X,T_k) = \sigma_k^2 \), \( k = 1, \ldots, K \). As in Section 3, we assume the existence of the moment generating function of \( \varepsilon \) when conditioned on \((X,T_k)\). We also assume that \( X \) is continuous with density \( f(x) \) and is supported on \([0,1]\). Let \( \theta_k, k = 0, \ldots, K \) be in increasing order, and such that treatment \( k \) is best in the interval \((\theta_{k-1}, \theta_k)\); we assume that each treatment is best in some open interval, or equivalently that the intervals are nonempty. Equations (2) and (5) for the regret and ideal regret imply immediately

\[
R(\pi_1, \ldots, \pi_K) = \sum_{k=1}^{K} \sum_{m=1}^{\hat{\theta}_m} P\left(\hat{g}_k(x) > \max_{\ell \neq k} \hat{g}_\ell(x)\right) [g_m(x) - g_k(x)] f(x) dx
\]

and the ideal regret is

\[
R_I(\pi_1, \ldots, \pi_K) = \sum_{k=1}^{K} \sum_{m=1}^{\hat{\theta}_m} \left( \int \prod_{l=1, \ldots, K, l \neq k} \Phi\left(\frac{z \xi_l(x) + \sqrt{n}[g_k(x) - g_l(x)]}{\xi_k(x)}\right) \varphi(z) dz \right) [g_m(x) - g_k(x)] f(x) dx.
\]

Similar to the results in Section 3, we give asymptotic results on the rate of convergence of the regret to the ideal regret \( R_I \) as well as the limit of the regret. Given allocation functions \( \pi_k(x) \), the distribution of the estimated regression coefficients \((\hat{\alpha}_k, \hat{\beta}_k)\) is approximately bivariate normal with means \((\alpha_k, \beta_k)\) and covariance

\[
\Sigma_k = \frac{\sigma_k^2}{\nu_k} \begin{pmatrix} \frac{\tau_k^2 + \mu_k^2}{\tau_k} & -\mu_k \tau_k^{-1} \\ -\mu_k \tau_k^{-1} & \frac{1}{\tau_k} \end{pmatrix},
\]

where we assume positivity of \( \nu_k := \int f(x)\pi_k(x) dx \), and denote the mean and variance of the covariate in group \( k \) by \( \mu_k := \int_0^1 x f_k(x) dx \) and \( \tau_k^2 := \int_0^1 (x - \mu_k)^2 f_k(x) dx \), respectively, \( k = 1, \ldots, K \). Parallel to Theorems 3.3 and 3.4 we have

**Theorem 4.1.** Under the assumptions in the beginning of Section 4.1 we have for any \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} n^{3/2-\varepsilon} |R(\pi_1, \ldots, \pi_K) - R_I(\pi_1, \ldots, \pi_K)| = 0.
\]

**Theorem 4.2.** Under the assumptions in the beginning of Section 4.1, we have

\[
n \lim_{n \to \infty} R(\pi_1, \ldots, \pi_K) = \sum_{m=1}^{K-1} \frac{V_m(\theta_m) f(\theta_m)}{2|\beta_{m+1} - \beta_m|},
\]

(15)
where \( V_m(x) = (1, x)(\Sigma_m + \Sigma_{m+1})(\frac{1}{x}) \).

Using (14) it is possible to write \( V_m(\theta_m) \) explicitly as follows

\[
V_m(\theta_m) = \frac{\sigma_m^2}{\nu_m} \left[ 1 + \frac{(\theta_m - \mu_m)^2}{\tau_m^2} \right] + \frac{\sigma_{m+1}^2}{\nu_{m+1}} \left[ 1 + \frac{(\theta_m - \mu_{m+1})^2}{\tau_{m+1}^2} \right].
\]

Theorem 4.2 implies that asymptotically the optimal allocation problem reduces to minimize a weighted average of the variances of \( \hat{\theta}_1, \ldots, \hat{\theta}_{K-1} \) (see the discussion after Theorem 3.4), which are the estimates of the intersection points. The weights depend on the \( \beta \)'s and on the density \( f \) at the intersection points. Note also that \( V_m(\theta_m) \) is the sum of the asymptotic variances of \( \hat{\alpha}_m + \hat{\beta}_m \theta_m \) and \( \hat{\alpha}_{m+1} + \hat{\beta}_{m+1} \theta_m \). Corollary 3.1 continues to hold for \( K \) treatments and thus, optimizing the ideal regret approximately optimizes the regret itself.

### 4.2 A lower bound for the asymptotic regret (15)

By (13), (14), and (15) (see also (3)) both the ideal and the asymptotic regret depend on the functions \( \pi_k(x) \) only via the quantities \( \nu_k, \mu_k, \tau_k^2, k = 1, \ldots, K \). Therefore, we can minimize (13) and (15) in these parameters to obtain a lower bound for the regret. Let \( \mu := \int_0^1 xf(x) \, dx \), \( \tau^2 := \int_0^1 (x - \mu)^2 f(x) \, dx \) and recall that \( f(x) = \sum_{k=1}^K \nu_k f_k(x) \). Then, by the relations for central moments of mixture distributions we have

\[
\sum_{k=1}^K \nu_k = 1, \quad \sum_{k=1}^K \nu_k \mu_k = \mu, \quad \sum_{k=1}^K \nu_k (\tau_k^2 + \mu_k^2) = \tau^2 + \mu^2. \tag{16}
\]

Minimizing (13) or (15) in \( \nu_k, \mu_k, \tau_k, k = 1, \ldots, K \) subject to the constraints (16) and \( \nu_k, \tau_k \geq 0, k = 1, \ldots, K \) yields a lower bound for the achievable regret. To obtain the minimum regret (instead of a lower bound) one needs to add additional constraints on \( \mu_k, \tau_k, k = 1, \ldots, K \) to restrict optimization to values for which there exist mixture components \( f_k \) with weighted sum \( f \) that assume these moments.

We now consider an example with uniform \( f \) and \( K = 3 \), and minimize the asymptotic regret (15). Since now \( f(x) \leq 1 \) the constraint \( f(x) = \sum_{k=1}^3 \nu_k f_k(x) \) implies that \( \nu_k f_k(x) \leq 1, k = 1, 2, 3, x \in [0, 1] \). By Lemma 1 below this implies that the variance of \( f_k(x), k = 1, 2, 3 \) is bounded from below by \( \nu_k^2/12 \).

As an example consider the setting where \( (\alpha_1, \alpha_2, \alpha_3) = (0.0, -0.1, -1.2), (\beta_1, \beta_2, \beta_3) = (0.2, 0.5, 2.0) \), and the residual variances \( \sigma_k^2, k = 1, 2, 3 \) are equal across treatment groups. Figure 2 shows the resulting scenario and optimized allocation probabilities (whose computation is explained below). Then, numerically (see details below) minimizing (15) in \( \nu_k, \mu_k, \tau_k, k = 1, \ldots, K \) subject to the constraints (16) as well as \( \tau_k^2 \geq \nu_k^2/12, k = 1, 2, 3 \) gives \( \nu_k = 0.346, 0.444, 0.210, \mu_k = 0.342, 0.512, 0.735, \tau_k^2 = 0.00997, 0.132, 0.00369 \), respectively, and the minimized value of (15) is 12.128. We note that \( \tau_k^2 = \nu_k^2/12, \) for \( k = 1, 3 \). Applying Lemma 1 for \( c = 1/\nu_k \) for \( k = 1, 3 \), the only density \( f_k(x) \) with mean \( \mu_k \)
Figure 2: The functions $g_k$ for the three treatments (left) and the optimal allocation probabilities minimizing (15) (as a stacked area chart, right) for the example in Section 4.2. For instance, for $x$’s under the black area, treatment 1 is assigned with probability $\pi_1(x) = 1$.

and variance $\tau^2_k$ and satisfying the constraint $f_k(x) \leq 1/\nu_k$, $x \in [0,1]$ is the uniform distributions on $[\mu_k - \nu_k/2, \mu_k + \nu_k/2]$ with densities $f_k(x) = 1_{[\mu_k-\nu_k/2, \mu_k+\nu_k/2]}(x)/\nu_k$. Furthermore, we set

$$f_2(x) = \frac{1 - \nu_1 f_1(x) - \nu_3 f_3(x)}{\nu_2}.$$  

Because the supports of $f_1, f_3$ are disjoint and $f_k(x) \leq 1/\nu_k$, $k = 1, 3$, $f_2(x)$ is a valid density. By construction, the mean and variance of $f_2(x)$ are $\mu_2, \tau^2_2$ and the three densities satisfy $f(x) = \sum_{k=1}^{3} \nu_k f_k(x)$. Thus, the resulting allocation probabilities $\pi_k(x) = f_k(x)/\nu_k$ achieve the lower bound computed above, and hence, assuming the minimization algorithm yielded the correct minimum (which we verified in several ways, including the fact that it is unique), it is an optimal allocation (see Figure 2 for a plot). It follows that in this case the above constraints suffice. Furthermore, as $f_1, f_3$ are unique, they are a unique optimal allocation (under the assumption that the minimization in $\nu_k, \mu_k, \tau_k, k = 1, \ldots, K$ gives a unique solution). It also follows that the optimal solution leads to a deterministic allocation.

Some comments: (1) To implement the numerical optimization of (15) subject to the constraints, we use the R command optim with the following reparametrization, which allows one to apply optimization algorithms for unrestricted optimization. Let $A_i = (a_{i,1}, a_{i,2}) \in \mathbb{R}^2$, $i = 1, 2, 3$. First we transform the vectors $A_i$ into proportions setting $a_{ik}' = \exp(a_{ik})/\sum_{m=1}^{3} \exp(a_{i,m})$, $i, k = 1, 2, 3$ where $a_{i,3} \equiv 0$, and set $\nu_k = a_{1k}', \mu_k = a_{2k}' \mu_k/\nu_k$, and

$$\tau^2_k = \nu_k^2 + a_{3,k}' \left( \mu^2 + \tau^2 - \sum_{m=1}^{3} (\nu_m \mu_m^2 + \nu_m^3/12) \right)$$
As long as the second summand remains positive, this parametrization ensures that \( r_k^2 \geq \nu_k^2/12 \) and that the constraints of (16) hold. To confirm the robustness of the solution, the optimization is performed repeatedly with randomly chosen initial conditions. (2) If the intersection points \( \theta_{k,k+1} \) of the lines \( g_k \) move closer together so will the intervals of \( x \)'s where all patients are allocated to Treatment 1 or Treatment 3 (the blue and black regions in Figure 2). For scenarios where they overlap, we can no longer use the above approach to obtain an optimal solution and the solutions seem to become more complex. This is the case, for example, for \( \alpha_k = 0.0, -0.1, -0.8, \beta_k = 0.2, 0.5, 2.0, \ k = 1, 2, 3 \).

**Lemma 1** Let \( c > 0 \). Among all continuous distributions \( f \) with (i) mean \( \mu \) such that (ii) \( f(x) \leq c \) for all \( x \in \mathbb{R} \), the uniform distribution on \( [\mu - 1/(2c), \mu + 1/(2c)] \) has the smallest variance. This minimum variance is \( 1/(c^2/12) \).

## 5 A clinical trial to determine personalized diets

To demonstrate our approach, we consider a study by Ebbeling et al. (2018) in which three diets were compared in a parallel group design. We fitted a model to the data of that trial and derived an alternative optimal design based on the estimated model parameters.

Patients were randomly assigned to three diet groups. The diets differ in their carbohydrate content, high (60%), moderate (40%) or low (20%) and were given for 20 weeks. The total number of subjects allocated to these three treatment groups was 54, 53 and 57, respectively, so the design is (almost) balanced. The main covariate is insulin secretion (insulin concentration 30 minutes after oral glucose). The primary outcome in the original trial was averaged total energy expenditure over two measurements, in the middle and the end of the trial. Total energy expenditure is measured by doubly labeled water; for details on this measure see Hills et al. (2014).

Of the 1685 subjects initially screened, 234 participated in a run-in phase which preceded the trial itself. Of these, 164 achieved a target of 12\% (± 2\%) weight loss and qualified for the trial, and were randomly assigned to one of the three diets.

To account for the cost of the diets, either in terms of money or in terms of inconvenience to the dieter, we consider the utility of the diets as an outcome variable which is defined as the energy expenditure minus the diet cost. We assume that the lower the carbon content of the diet, the higher the cost (see Hagberg et al. (2019) for a similar definition) but otherwise our choice of the cost for this example is arbitrary. On the basis of Figure 4 in Ebbeling et al. (2018), we assume that under treatment \( T_k, k = 1, 2, 3 \),

\[
\text{utility} = \alpha_k + \beta_k X + \varepsilon_k - \text{cost}_k, \tag{17}
\]
where \( \text{cost}_k = 0, 150, 300 \) for \( k = 1, 2, 3 \), respectively, \( X \) is insulin secretion and its distribution is \( \text{Gamma}(3.12, 0.02) \), and the standard deviations of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are 190, 150, 130, respectively. The design we propose is based on assumed values of \( \alpha_k \), which are 40, -80, -240, and of \( \beta_k \), which are -0.8, 0.1, and 0.8 for \( k = 1, 2, 3 \), respectively. These values and the parameters of the gamma distribution of \( X \) and the standard deviation of \( \varepsilon \)'s are all based on Figure 4 of Ebbeling et al. (2018), which plots the energy expenditure as a function of the insulin secretion for the three diets in the data. In particular, the parameters of the Gamma distribution are chosen to match empirical quantiles of the \( X \)'s.

In order to numerically find the optimal allocation probabilities \( \pi_k(x) \) that minimize the regret, we use a certain parameterization. Let \( \mathbf{a} = (a_0, \ldots, a_m) \), and set \( h_\mathbf{a}(x) = \exp(\sum_{j=0}^{m} a_j x^j) \). Furthermore, let \( \mathbf{A} \) denote a \( 2 \times (m+1) \) matrix of real numbers. Let \( \pi_k(x) = \frac{h_{\mathbf{A}_k}(x)}{\sum_{j=0}^{m} h_{\mathbf{A}_j}(x)}, \quad k = 1, 2, 3 \), where \( \mathbf{A}_k \) denotes the \( k \)-th row of \( \mathbf{A} \) and we set \( h_{\mathbf{A}_3}(x) \equiv 1 \). Now we can approximately minimize the (ideal) regret by plugging \( \pi_k(x) \) into (13) and minimizing with respect to \( \mathbf{A} \). In this example we chose \( m = 4 \) and use the \( \text{R} \) function \( \text{optim} \) for the numerical optimization.

Figure 3(a) plots the regression lines (17) for the three diets and the assumed distribution of the covariate \( x \). The high, moderate, and low carbohydrate diets are optimal when \( X \) belongs to the intervals \((0, 133), (133, 229), (229, \infty)\), respectively. The difference between the treatments is more pronounced for extreme values of \( X \). Hence, people with very small or very large \( X \) would benefit more from a personalized treatment choice compared to those with medium values of \( X \).

Figures 3(b)-(d) present the optimal allocation probabilities, as a stacked area chart, when \( n = 164, 1000, \infty \). When the sample size is larger, the optimal allocation rule is closer to being deterministic. For \( n = 164 \), which was the actual trial size, the optimal trial allocation rule tends to allocate subjects to \( T_3 \) (respectively, \( T_2 \)) for large (respectively, small) values of \( X \). Compared to equal allocation, which was the actual design used, there is a reduction of about 11% of the regret. A saving of 11% may be quite significant if the treatments are applied to a large number of future patients. Furthermore, the regret of the balanced design with \( n = 164 \) can be achieved under the optimal design with \( n = 143 \), which represents a reduction of 13% of the sample size. Taking into account that in this experiment only about 10% of recruited patients passed the screening process, a saving of about 20 subjects amounts to a reduction of about 200 subjects that would have to be recruited and pretested.

To investigate if the reduction in the regret is due to unbalanced allocation or the covariate dependent allocation, we also computed the regret for the optimal unbalanced allocation that is restricted to allocation probabilities that do not depend on the covariate. We refer to this design as \textit{unbalanced}. This can be achieved as above setting \( m = 0 \). We found numerically the optimal unbalanced design for \( n = 164 \) leads to reduction of the regret by about 8%, whereas optimizing with allocation functions that depend on the
covariate, leads to a reduction by an additional 3%. For larger sample sizes, it appears that the gain by optimized covariate dependent allocation probabilities increases (see Table 1). As the algorithm only gives an approximate optimal solution, we also used the approach in Section 4.2 to compute a lower bound for the limit $nR_I$ for $n \to \infty$. The obtained lower bound of 735.1 is close to 738.2, the corresponding regret obtained by the derived allocation function depicted in Figure 3d.

It should be noticed that all these designs are based on the true values of the parameters and therefore require a preliminary study, or an adaptive design.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$nR_I$</th>
<th>% reduction</th>
<th>optimized allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>optimal</td>
<td>unbalanced</td>
</tr>
<tr>
<td>164</td>
<td>842.8</td>
<td>10.9</td>
<td>7.9</td>
</tr>
<tr>
<td>1000</td>
<td>829.6</td>
<td>13.1</td>
<td>4.4</td>
</tr>
<tr>
<td>$\infty$</td>
<td>738.2</td>
<td>17.1</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Table 1: Diets example: $nR_I$, the reductions of the regret of the optimal design compared to a balanced design, the reductions of the regret of a design with optimized allocation ratios only (where allocation probabilities do not depend on $X$) compared to the balanced design, and the optimized allocation probabilities in percent (for the design where allocation probabilities do not depend on $X$).
Figure 3: Figure (a) plots the regression models (17) and the assumed density of the covariate. Figures (b)-(d) present the optimal allocation probabilities (as a stacked area chart) when $n = 164, 1000, \infty$. 
A Appendix: Proofs

Proof of Lemma 3.2

We use the convexity relation \( \lambda A^{-1} + (1 - \lambda)B^{-1} \geq [\lambda A + (1 - \lambda)B]^{-1} \) (Moore, 1973) where \( A \) and \( B \) are positive definite matrices. We have

\[
\Sigma_1 + \Sigma_2 = \sigma_1 \left( \frac{\nu_1 Q_1}{\sigma_1} \right)^{-1} + \sigma_2 \left( \frac{\nu_2 Q_2}{\sigma_2} \right)^{-1}
\geq \left( \frac{\sigma_1}{\sigma_1 + \sigma_2} \right) \left[ \frac{\nu_1 Q_1}{\sigma_1} + \frac{\nu_2 Q_2}{\sigma_2} \right]^{-1} = (\sigma_1 + \sigma_2)^2 Q^{-1},
\]

where in the last equality we used the fact that \( \nu_1 Q_1 + \nu_2 Q_2 = Q \). Equality holds when the two matrices to the left of the \( \geq \) sign are equal, which happens when \( \pi_1(x) = \frac{\sigma_1}{\sigma_1 + \sigma_2} \) and \( \pi_2(x) = \frac{\sigma_2}{\sigma_1 + \sigma_2} \).

Proof of Theorem 3.3

We start with the case of \( p = 1 \). It is enough to show (as the other part of the integral, from \( \theta \) to \( 1 \), is symmetric) that

\[
n^{3/2-\varepsilon} \int_0^\theta \left| P(\hat{g}_2(x) > \hat{g}_1(x)) - \Phi \left( -\frac{\sqrt{n}[g_1(x) - g_2(x)]}{V(x)} \right) \right| |g_1(x) - g_2(x)| f(x) dx \to 0. \tag{18}
\]

The idea of the proof is quite simple. For \( x \in \theta \pm 1/\sqrt{n} \) the quantity in absolute value converges to zero, and the range of the integral and \( g_1(x) - g_2(x) \) are of order \( 1/\sqrt{n} \) each. For other \( x \)'s both the probability and \( \Phi \) in the integrand are exponentially small in \( n \).

We will prove (18) by showing that

\[
n^{3/2-\varepsilon} \int_0^\theta \left| P(\hat{g}_2(x) > \hat{g}_1(x)) - E \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{S(x)} \right) \right| |g_1(x) - g_2(x)| f(x) dx \to 0 \tag{19}
\]

\[
n^{3/2-\varepsilon} \int_0^\theta \left| E \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{S(x)} \right) - \Phi \left( -\frac{\sqrt{n}[g_1(x) - g_2(x)]}{\sqrt{V(x)}} \right) \right| |g_1(x) - g_2(x)| f(x) dx \to 0, \tag{20}
\]

where

\[
S^2(x) = n(1, x) \left\{ \sigma_1^2 \left( \sum_{i \in T_1} \left( \begin{array}{c} 1 \\ X_i \end{array} \right) \right)^{-1} + \sigma_2^2 \left( \sum_{i \in T_2} \left( \begin{array}{c} 1 \\ X_i \end{array} \right) \right)^{-1} \right\} \left( \begin{array}{c} 1 \\ x \end{array} \right),
\]

and the expectation in (19) and (20) is with respect to \( X_1, \ldots, X_n \) which appear in \( S^2(x) \) and the random sets \( T_k \). We start with (20). Since \( \theta \) is the intersection point, \( \alpha_1 + \beta_1 \theta = \alpha_2 + \beta_2 \theta \); therefore,
\( g_1(x) - g_2(x) = \alpha_1 + \beta_1 x - (\alpha_2 + \beta_2 x) = (\beta_2 - \beta_1)(\theta - x) \). Hence, (20) can be written as

\[
 n^{3/2 - \epsilon} \int_0^\theta \left| \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{S(x)} \right) - \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V(x)}} \right) \right| (\beta_2 - \beta_1)(\theta - x)f(x)dx \to 0.
\]

Fix \( c_n = n^{\epsilon/5} \). We will show that

\[
 n^{3/2 - \epsilon} \int_0^{\theta - c_n / \sqrt{n}} \left| \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{S(x)} \right) - \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V(x)}} \right) \right| (\beta_2 - \beta_1)(\theta - x)f(x)dx \to 0,
\]

and that

\[
 n^{3/2 - \epsilon} \int_0^\theta \left| \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{S(x)} \right) - \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V(x)}} \right) \right| (\beta_2 - \beta_1)(\theta - x)f(x)dx \to 0.
\]

We show (22) by arguing that the probabilities in (22) are exponentially small. With notation defined in Section 2 we have that

\[
 V(x) \leq (1 + x^2) \left( \frac{\sigma_1^2}{\nu_1 \lambda_{\text{min}}(Q_1)} + \frac{\sigma_2^2}{\nu_2 \lambda_{\text{min}}(Q_2)} \right) \leq C,
\]

for a constant \( C > 0 \) (since \( x \) is bounded). With the bound \( \Phi(-t) \leq \exp(-t^2/2) \) for \( t > 1 \) and \( (\theta - x) \sqrt{n} \leq c_n \) we have

\[
 \Phi \left( -\frac{(\beta_2 - \beta_1)(\theta - x)\sqrt{n}}{\sqrt{V(x)}} \right) \leq \exp \left( -\frac{(\beta_2 - \beta_1)^2 c_n^2}{2C} \right),
\]

which shows that the second normal probability \( \Phi(\cdot) \) in (22) is exponentially small. We now argue that the expectation in (22) is also exponentially small. Consider the event

\[
 A_n := \left\{ \lambda_{\text{min}} \left( \frac{1}{n} \sum_{i \in T_1} \left( \frac{1}{X_i} \right)(1, X_i) \right) \leq \nu_1 \frac{\lambda_{\text{min}}(Q_1)}{2} \right\}
\]

\[
 \cup \left\{ \lambda_{\text{min}} \left( \frac{1}{n} \sum_{i \in T_2} \left( \frac{1}{X_i} \right)(1, X_i) \right) \leq \nu_2 \frac{\lambda_{\text{min}}(Q_2)}{2} \right\}.
\]

Writing \( \frac{1}{n} \sum_{i \in T_1} \left( \frac{1}{X_i} \right)(1, X_i) = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{X_i} \right)(1, X_i)(i \in T_1) \), a sum of iid terms, we can apply Theorem 5.1 of Tropp (2012) and the union bound to conclude that \( P(A_n) \leq C \exp(-Cn) \) for some constant \( C > 0 \).
Going back to (22), write

\[
E\Phi\left(-\sqrt{n}(\beta_2 - \beta_1)(\theta - x)\right) \quad S(x) \\
= E\Phi\left(-\sqrt{n}(\beta_2 - \beta_1)(\theta - x)\right) I(A_n) + E\Phi\left(-\sqrt{n}(\beta_2 - \beta_1)(\theta - x)\right) I(A_n^c) \\
\leq P(A_n) + E\Phi\left(-\sqrt{n}(\beta_2 - \beta_1)(\theta - x)\right) I(A_n^c). 
\]

When \( A_n^c \) occurs, the denominator \( S^2(x) \) is bounded and the same argument as above shows that this probability is exponentially small. We conclude that both probabilities in (22) are exponentially small, which implies that (22) is correct.

We now prove (23). By the mean value theorem, for \( x \in (\theta - c_n/\sqrt{n}, \theta) \) and for some \( V^*(x) \) between \( S^2(x) \) and \( V(x) \), we have for some \( C > 0 \)

\[
\left| \Phi\left(-\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{S^2(x)}}\right) - \Phi\left(-\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V(x)}}\right) \right| \\
= |S^2(x) - V(x)| \varphi\left(\frac{-\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V^*(x)}}\right) \frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{2} (V^*(x))^{-3/2} \\
\leq C |S^2(x) - V(x)| c_n (\min\{S^2(x), V(x)\})^{-3/2}, \text{ and} \\
\{S^2(x) - V(x)\} = (1, x) \left\{ \frac{\sigma^2}{\nu_1} (M_1^{-1} - Q_1^{-1}) + \frac{\sigma^2}{\nu_2} (M_2^{-1} - Q_2^{-1}) \right\} \left( \begin{array}{c} 1 \\ x \end{array} \right), \quad (25)
\]

where \( M_k = \frac{1}{w_k} \sum_{i \in T_k} \left( \frac{1}{X_i} \right) X_i \), \( k = 1, 2 \). We have that \( M_1^{-1} - Q_1^{-1} = M_1^{-1} (Q_1 - M_1) Q_1^{-1} \), implying that the difference of the inverses is small when the difference of the matrices is small and the inverse is bounded. Define the event \( \bar{A}_n := \{ \max\{\|Q_1 - M_1\|_\infty, \|Q_2 - M_2\|_\infty\} \geq c_n/\sqrt{n} \} \), where \( \| \cdot \|_\infty \) denotes the maximum (entry-wise) norm. By Hoeffding’s inequality (since the random variables are bounded), \( P(\bar{A}_n) \leq C \exp(-Cc_n^2) \) (which is exponentially small). Therefore, as before, this event can be ignored. On the complement of \( \bar{A}_n \), \( M_1^{-1}, M_2^{-1}, S^2(x) \) are all bounded, and by similar arguments \( S^2(x) \) and \( V(x) \) are bounded below. Hence, when \( \bar{A}_n^c \) occurs,

\[
\left| \Phi\left(-\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{S(x)}\right) - \Phi\left(-\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V(x)}}\right) \right| \leq C \frac{c_n^2}{\sqrt{n}}. 
\]
Therefore the integral in (23) is bounded by

\[ Cn^{1-\varepsilon}c_n^3 \int_{\theta-c_n/\sqrt{n}}^{\theta} (\theta - x) f(x) dx \leq Cn^{1/2-\varepsilon}c_n^3 \int_{\theta-c_n/\sqrt{n}}^{\theta} f(x) dx \leq Cc_n^4/n^\varepsilon = n^{-\varepsilon/5} \to 0, \]

which completes the proof of (23) and hence of (20).

We now show (19) by dividing the integral into two ranges: (a) \([\theta - c_n/\sqrt{n}, \theta]\), and (b) \([0, \theta - c_n/\sqrt{n}]\).

We have that

\[
P(\hat{g}_2(x) > \hat{g}_1(x)) = P\left( \left[ \left( \hat{\alpha}_1 - \alpha_1 \right) / (\hat{\beta}_1 - \beta_1) - \left( \hat{\alpha}_2 - \alpha_2 \right) / (\hat{\beta}_2 - \beta_2) \right] \cdot \left( \frac{1}{x} \right) > (\beta_2 - \beta_1)(x - \theta) \right)
\]

\[
= P\left( \left[ \frac{M_1^{-1}}{n \nu_1} \sum_{i \in T_1} \left( \varepsilon_i / X_i \right) - \frac{M_2^{-1}}{n \nu_2} \sum_{i \in T_2} \left( \varepsilon_i / X_i \right) \right] \cdot \left( \frac{1}{x} \right) > (\beta_2 - \beta_1)(x - \theta) \right)
\]

\[
= P\left( \sum_{i=1}^{n} a_i \varepsilon_i > (\beta_2 - \beta_1)(x - \theta) \right),
\]

where \(a_i = \begin{cases} 
(1, X_i) \frac{M_1^{-1}}{n \nu_1} (1/ \varepsilon_i) & i \in T_1, \\
-(1, X_i) \frac{M_2^{-1}}{n \nu_2} (1/ \varepsilon_i) & i \in T_2.
\end{cases}\)

Notice that \(Var(\varepsilon_i)\) is equal to \(\sigma_1^2\) if \(i \in T_1\) and to \(\sigma_2^2\) if \(i \in T_2\). We are going to condition on \(X_1, \ldots, X_n\) obtained in the experiment and their allocation, denoted together by \(D\). Then \(Var(\sum_{i=1}^{n} a_i \varepsilon_i | D) = S^2(x)/n\), where \(S^2(x)\) is defined in (21).

We aim to apply the Berry Esseen Theorem, for which we need some bounds. The values \(na_i\) are bounded below since the matrices \(M_k\) are bounded above. In order to obtain upper bounds, recall the event \(A_n\) defined in (24). On the set \(A_n^c\) we have that \(na_i\) are bounded above. Since \(P(A_n)\) is exponentially small, we can ignore it, and assume that \(na_i\) are bounded with probability one. We can apply the Berry-Esseen Theorem for the non-identically distributed case (for a convenient reference see Chen et al. (2010) (3.27)) to \(\sum_{i=1}^{n} \frac{\sqrt{na_i}}{S^2(x)} \varepsilon_i \equiv \sum_{i=1}^{n} V_i\) where \(\sum_{i=1}^{n} Var(V_i | D) = 1\) and \(E(|V_i|^3 | D) \leq \max\{na_i/S(x)E(|\varepsilon_i|^3 | D) : 1 \leq i \leq n\} n^{-3/2}\) to obtain by unconditioning on \(D\)

\[
\left| P\left( \sum_{i=1}^{n} a_i \varepsilon_i > (\beta_2 - \beta_1)(x - \theta) \right) - E \Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(x - \theta)}{S(x)} \right) \right| \leq C/\sqrt{n}.
\]
Therefore,

\[ n^{3/2-\varepsilon} \int_{\theta-c_n/\sqrt{n}}^{\theta} \left| P(\tilde{g}_2(x) > \tilde{g}_1(x)) - E\Phi \left( -\frac{\sqrt{n}(\beta_2 - \beta_1)(\theta - x)}{S(x)} \right) \right| (\beta_2 - \beta_1)(\theta - x)f(x)dx \]

\[ \leq n^{3/2-\varepsilon} C \frac{C}{\sqrt{n}} \int_{\theta-c_n/\sqrt{n}}^{\theta} (\beta_2 - \beta_1)(\theta - x)f(x)dx \leq n^{3/2-\varepsilon} C \frac{c_n}{\sqrt{n}} \int_{\theta-c_n/\sqrt{n}}^{\theta} f(x)dx \]

\[ \leq C \frac{c_n^2}{n^\varepsilon} = C \frac{n^{2\varepsilon/5}}{n^\varepsilon} \to 0. \quad (26) \]

By a standard large deviation bound, the integral of (19) from 0 to \( \theta - c_n/\sqrt{n} \) multiplied by \( n^{3/2-\varepsilon} \) is easily shown to be of order \( n^{3/2-\varepsilon}O(exp(-c_n/\sqrt{n})) \) \( \to 0 \). This proves the theorem for covariates in \( \mathbb{R} \) \( (p=1) \).

We now consider the case of covariates in \( \mathbb{R}^p \) for \( p \geq 2 \). Suppose, without loss of generality, that \( \beta_{2,1} > \beta_{1,1} \). For given \( x_{-1} = (x_2, \ldots, x_p) \) define \( \theta_1(x_{-1}) := \frac{\alpha_1 - \alpha_2 + (\beta_{1,1} - \beta_{2,1})x_{-1}}{\beta_{2,1} - \beta_{1,1}} \). For \( x \) such that \( \theta_1(x_{-1}) \notin [0, 1] \) the probability of making a mistake (i.e., \( \tilde{g}_2(x) - \tilde{g}_1(x) \) has the wrong sign) is exponentially small. Therefore,

\[
R(\pi_1, \pi_2) = \int_{[0,1]^{p-1}} \int_0^{\theta_1} P(\tilde{g}_2(x) > \tilde{g}_1(x)) (g_1(x) - g_2(x))f(x)dx_1 I(\theta_1 \in [0, 1])dx_{-1} \\
+ \int_{[0,1]^{p-1}} \int_{\theta_1}^1 P(\tilde{g}_2(x) > \tilde{g}_1(x)) (g_1(x) - g_2(x))f(x)dx_1 I(\theta_1 \in [0, 1]) dx_{-1} + a_n,
\]

where \( a_n \) is exponentially small. By a slight variation on the above one-dimensional case, the inner integral equals

\[
\int_0^1 \Phi \left( \frac{-\sqrt{n}|g_2(x) - g_1(x)|}{V(x)} \right) |g_2(x) - g_1(x)|f(x)dx_1 I(\theta_1 \in [0, 1]) + o(1/n^{3/2-\varepsilon}), \quad (27)
\]

where the error term \( o \) is uniform in \( x_{-1} \). Taking the outer integral, the theorem follows.

**Proof of Theorem 3.4**

As computed two lines below (21), \( g_1(x) - g_2(x) = (\beta_2 - \beta_1)(\theta - x) \). We will show that for continuous \( t \)

\[
\lim_{t \to \infty} t \int_0^\theta \Phi \left( \frac{-\sqrt{t}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V(x)}} \right) (\beta_2 - \beta_1)(\theta - x)f(x)dx = \frac{V(\theta)f(\theta)}{4(\beta_2 - \beta_1)}. \quad (28)
\]

The integral from \( \theta \) to 1 is similar, yielding the desired limit. By L’Hôpital’s rule the limit in (28) equals

\[
\lim_{t \to \infty} \frac{t^2}{2} \int_0^\theta \varphi \left( \frac{\sqrt{t}(\beta_2 - \beta_1)(\theta - x)}{\sqrt{V(x)}} \right) (\beta_2 - \beta_1)(\theta - x) \frac{1}{\sqrt{V(x)}} \sqrt{t}(\beta_2 - \beta_1)(\theta - x)f(x)dx.
\]
Substitution of $x$ by $y = (\theta - x)\sqrt{t}$ in the integral yields

$$
\lim_{t \to \infty} \frac{1}{2} \int_0^{\theta \sqrt{t}} \varphi \left( \frac{(\beta_2 - \beta_1)y}{\sqrt{V(\theta - y/\sqrt{t})}} \right) \left( \frac{(\beta_2 - \beta_1)^2}{\sqrt{V(\theta - y/\sqrt{t})}} \right) y^2 f(\theta - y/\sqrt{t}) dy.
$$

The limit is

$$
\frac{1}{2} \int_0^\infty \varphi \left( \frac{(\beta_2 - \beta_1)y}{\sqrt{V(\theta)}} \right) \left( \frac{(\beta_2 - \beta_1)^2}{\sqrt{V(\theta)}} \right) y^2 f(\theta) dy.
$$

Substitution of $y$ with $z = \frac{(\beta_2 - \beta_1)y}{\sqrt{V(\theta)}}$ in the integral yields the claimed limit $\frac{f(\theta)V(\theta)}{2(\beta_2 - \beta_1)} \int_0^\infty \varphi(z) z^2 dz = f(\theta)V(\theta) \frac{\pi}{4(\beta_2 - \beta_1)}$. \hfill \Box

**Proof of Theorem 3.5**

Consider the inner integral in (27). By Theorem 3.4, the limit of the inner integral times $n$ is

$$
V(\theta_1, x_{-1}) f(\theta_1, x_{-1})
\frac{2(\beta_{2,1} - \beta_{1,1})}{4(\beta_2 - \beta_1)}.
$$

By taking the outer integral the result follows. \hfill \Box

**The limit of the regret for polynomial regression**

Suppose that $g_1(x) = \alpha_1 + (x, \ldots, x^J)^t \beta_1$ and $g_2(x) = \alpha_2 + (x, \ldots, x^J)^t \beta_2$. Let $\theta$ be an intersection point, i.e., $g_1(\theta) = g_2(\theta)$. We have

$$
g_2(x) - g_1(x) = g_2(x) - g_2(\theta) - [g_1(x) - g_1(\theta)] = \eta^t (\beta_2 - \beta_1),
$$

where $\eta := (x - \theta, x^2 - \theta^2, \ldots, x^J - \theta^J)^t$. The following proposition is parallel to Theorem 3.4 (in the one-dimensional case) and provides the limit of the regret; see (12). The proof is similar, although some modifications are required.

**Proposition A.1.** For continuous $t$

$$
\lim_{t \to \infty} t \int_0^\theta \Phi \left( \frac{-\sqrt{t}(\beta_2 - \beta_1)^t \eta}{\sqrt{V(x)}} \right) (\beta_2 - \beta_1)^t \eta f(x) dx = \frac{f(\theta)V(\theta)}{4|\beta_2 - \beta_1|^t \zeta}, \quad (29)
$$

where $\zeta := (1, 2\theta, \ldots, J\theta^{J-1})^t$. 23
Proof. By L'Hôpital's rule the limit in (29) equals

\[
\lim_{t \to \infty} \frac{t^2}{2} \int_0^\theta \varphi \left( \frac{\sqrt{t}(\beta_2 - \beta_1)^t\eta}{\sqrt{V(x)}} \right) \frac{(\beta_2 - \beta_1)^t\eta}{\sqrt{V(x)}} \frac{1}{\sqrt{t}} (\beta_2 - \beta_1)^t\eta f(x) dx.
\]

Substituting \( y = (\theta - x)\sqrt{t} \) we get

\[
\sqrt{t}\eta = \sqrt{t} \begin{pmatrix} x - \theta \\ x^2 - \theta^2 \\ \vdots \\ x^J - \theta^J \end{pmatrix} = \sqrt{t} \begin{pmatrix} x - \theta \\ (x - \theta)(x + \theta) \\ \vdots \\ (x - \theta)(x^{J-1} + x^{J-2}\theta + \ldots + \theta^{J-1}) \end{pmatrix} = y\tilde{\eta},
\]

where

\[
\tilde{\eta} := (1, x + \theta, \ldots, x^{J-1} + x^{J-2}\theta + \ldots + \theta^{J-1})^t
\]

\[
= (1, 2\theta - y/\sqrt{t}, \ldots, (\theta - y/\sqrt{t})^{J-1} + (\theta - y/\sqrt{t})^{J-2}\theta + \ldots + \theta^{J-1})^t.
\]

The integral reads

\[
\lim_{t \to \infty} \frac{1}{2} \int_0^{\theta\sqrt{t}} \varphi \left( y \frac{(\beta_2 - \beta_1)^t\tilde{\eta}}{\sqrt{V(\theta - y/\sqrt{t})}} \right) \frac{y^2[(\beta_2 - \beta_1)^t\tilde{\eta}]^2}{\sqrt{V(\theta - y/\sqrt{t})}} f(\theta - y/\sqrt{t}) dy.
\]

The limit is

\[
\frac{1}{2} \int_0^{\infty} \varphi \left( \frac{y(\beta_2 - \beta_1)^t\zeta}{\sqrt{V(\theta)}} \right) \frac{[y(\beta_2 - \beta_1)^t\zeta]^2}{\sqrt{V(\theta)}} f(\theta) dy.
\]

Substitution of \( y \) with \( z = \frac{y(\beta_2 - \beta_1)^t\zeta}{\sqrt{V(\theta)}} \) yields

\[
\frac{f(\theta)V(\theta)}{2!(\beta_2 - \beta_1)^t\zeta} \int_0^{\infty} \varphi(z) z^2 dz = \frac{f(\theta)V(\theta)}{4!(\beta_2 - \beta_1)^t\zeta},
\]

which completes the proof. \( \square \)

For the proofs of Theorems 4.1 and 4.2 we need the lemma below and some notation. For every \( m \) and \( k \), let \( I_{m,k} := \int_{\theta m^{-1}}^{\theta m} P(\hat{g}_k(x) > \max_{\ell \neq k} \hat{g}_\ell(x)) [g_m(x) - g_k(x)] f(x) dx \). We then have \( R(\pi_1, \ldots, \pi_K) = \sum_{k=1}^{K} \sum_{m=1}^{K} I_{m,k} \).

Lemma A.2. Under the assumptions in the beginning of Section 4.1, the integral \( I_{m,k} \) is exponentially
small for \( k \neq m - 1, m + 1 \), and for every \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} n^{3/2-\varepsilon} \left| I_{m,m-1} - \int_{\theta_{m-1}}^{\theta_{m-1}+\epsilon_{n}} \Phi \left( \frac{-\sqrt{n}(g_m(x)-g_{m-1}(x))}{\sqrt{V_{m-1}(x)}} \right) [g_m(x) - g_{m-1}(x)] f(x)dx \right| = 0, \tag{30}
\]

\[
\lim_{n \to \infty} n^{3/2-\varepsilon} \left| I_{m,m+1} - \int_{\theta_{m-1}}^{\theta_{m}} \Phi \left( \frac{-\sqrt{n}(g_m(x)-g_{m+1}(x))}{\sqrt{V_m(x)}} \right) [g_m(x) - g_{m+1}(x)] f(x)dx \right| = 0,
\]

where \( c_n = n^{\varepsilon/5} \) and \( V_m(x) = (1, x) (\Sigma_m + \Sigma_{m+1})^{-1}(x) \). Moreover, for every \( m \) and \( k \)

\[
n^{3/2-\varepsilon} \left| I_{m,k} - \int_{\theta_{m-1}}^{\theta_{m}} \left( \int \prod_{l=1,\ldots,K,l\neq k} \Phi \left( \frac{z\xi_k(x) + \sqrt{n}[g_k(x) - g_l(x)]}{\xi_l(x)} \right) \varphi(z)dz \right) [g_m(x) - g_k(x)] f(x)dx \right| \to 0,
\]

as \( n \to \infty \).

**Proof of Lemma A.2**

The proof of the first claim in Lemma A.2 is similar to that of Theorem 3.3. Here is a sketch. For \( x \in I_{m,k} \) and bounded away from \( \theta_{m-1} \) and \( \theta_m \), and \( k \neq m \) both the quantities

\[
P \left( \hat{g}_k(x) > \max_{\ell \neq k} \hat{g}_\ell(x) \right) \quad \text{and} \quad \int \prod_{l=1,\ldots,K,l\neq k} \Phi \left( \frac{z\xi_k(x) + \sqrt{n}[g_k(x) - g_l(x)]}{\xi_l(x)} \right) \varphi(z)dz
\]

are exponentially small (for \( k = m \) the regret is zero). The same holds true for every \( x \) and when \( k \neq m - 1, m + 1 \).

We now prove (30) concerning \( I_{m,m-1} \) (where \( m > 1 \)). The other relation for \( I_{m,m+1} \) is similar. Fix \( c_n = n^{\varepsilon/5} \) and consider \( x \) such that \( x \in (\theta_{m-1}, \theta_{m-1} + c_n/\sqrt{n}) \). For such \( x \) we have by a standard large deviations argument that

\[
P \left( \hat{g}_{m-1}(x) > \max_{\ell \neq m-1} \hat{g}_\ell(x) \right) = P(\hat{g}_{m-1}(x) > \hat{g}_m(x)) + a_n, \quad \text{and} \quad \int \prod_{l=1,\ldots,K,l\neq m-1} \Phi \left( \frac{z\xi_{m-1}(x) + \sqrt{n}[g_{m-1}(x) - g_l(x)]}{\xi_l(x)} \right) \varphi(z)dz = \Phi \left( \frac{-\sqrt{n}[g_m(x) - g_{m-1}(x)]}{\sqrt{V_{m-1}(x)}} \right) + b_n,
\]

where \( a_n \) and \( b_n \) are exponentially small (uniformly in \( x \)). By a Berry-Esseen type bound applied to the first part of the integrand, and the smallness of the other part for \( x \) is the given range of the integral, and the smallness of the range itself, as in (26), we obtain

\[
\int_{\theta_{m-1}}^{\theta_{m-1}+\epsilon_{n}} \left[ P(\hat{g}_{m-1}(x) > \hat{g}_m(x)) - \Phi \left( \frac{-\sqrt{n}[g_m(x) - g_{m-1}(x)]}{\sqrt{V_m(x)}} \right) \right] [g_m(x) - g_{m-1}(x)] f(x)dx = o(1/n^{3/2-\varepsilon}),
\]

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and (30) follows. This, together with (31), implies the last part of the theorem when \( k = m - 1 \).

**Proof of Theorem 4.1**

This follows directly from the last part of Lemma A.2 and the fact that \( R(\pi_1, \ldots, \pi_K) = \sum_{k,m=1}^{K} I_{m,k} \).

Note that we can replace that latter sum by \( \sum_{m=1}^{K} I_{m,m-1} \) since \( I_{m,k} \) is exponentially small for \( k \neq m - 1, m + 1 \).

**Proof of Theorem 4.2**

The first part of Lemma A.2 implies that

\[
\sum_{k=1}^{K} \sum_{m=1}^{K} I_{m,k} = \sum_{m=1}^{K-1} \int_{\theta_m - \frac{\sqrt{n}}{\sqrt{V_{m-1}(x)}}}^{\theta_m + \frac{\sqrt{n}}{\sqrt{V_{m-1}(x)}}} \Phi \left( \frac{-\sqrt{n}|g_m(x) - g_{m-1}(x)|}{\sqrt{V_{m-1}(x)}} \right) |g_m(x) - g_{m-1}(x)| f(x) dx + o(1/n).
\]

The limit of the latter integrals can be calculated using Theorem 3.4 implying the result.

**References**


Bastani, H. and M. Bayati (2015). Online decision-making with high-dimensional covariates. *Available at SSRN 2661896*.


