Dynamic Allocation and Pricing : A Mechanism Design Approach*

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1 Introduction

Dynamic allocation and pricing problems appear in numerous frameworks such as the retail of seasonal/style goods, the allocation of fixed capacities in the travel and leisure industries (e.g., airlines, hotels, rental cars, holiday resorts), the allocation of a fixed inventory of equipment in a given period of time (e.g., equipment for medical procedures, bandwidth or advertising space in online applications), and the assignment of personnel to incoming tasks. Although dynamic pricing is a very old technique (think about haggling in a bazaar!), modern Revenue Management (RM) techniques started with US Airline Deregulation Act of 1978 (see McAfee and te-Velde [22]). A major academic textbook is *The Theory and Practice of Revenue Management* by K. T. Talluri and G.J. van Ryzin [30]. According to these authors, the basic RM issues are:

1) Quantity decisions: How to allocate capacity/output to different segments, products or channels? When to withhold products from the market?

2) Structural decisions: Which selling format to choose (posted prices, negotiations, auctions, etc..)? Which features to use for a particular format (segmentation, volume discounts, bundling, etc..)?

3) Pricing decisions: How to set posted prices, reserve prices? How to price differentiate? How to price over time? How to markdown over life time?

Broadly speaking, all above questions deal in fact with issues treated in the *Auction/Mechanism Design* literature (see for example Milgrom's [25] textbook). Nevertheless, mechanism design has not been the tool of choice in RM: instead, most papers have focused on analyzing properties of restricted classes (sometime intuitive, sometimes rather ad-hoc) of allocation/pricing schemes. One possible explanation for this gap is that the classical auction/mechanism

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design literature had a strong focus on static models while the emphasis in RM is on dynamics.

Thus, what is necessary for a modern theory of RM is a blend between the elegant *dynamic models* from the operations research, management science, computer science (with historical focus on grand, centralized optimization and/or "ad-hoc" mechanisms), and the classical mechanism design literature (with historical focus on information/incentives in static settings). Such a blend will be fruitful for numerous applications. Recently, this challenge has been addressed by a more or less systematic body of work appearing under the heading of *Dynamic Mechanism Design.*[1] Here we very briefly illustrate this approach, as reflected in our recent work. The present illustration and analysis are based on an elegant, early model due to Derman, Lieberman and Ross [9] who studied a dynamic version of (assortative) matching of objects to agents.

2 Becker vs. Derman-Lieberman-Ross

There are n agents who arrive sequentially, one agent per period. Periods are counted backwards, so that the last period is period 1 and the first period is period n. There are m items. Each item $i = 1, .., m$ is characterized by a "quality" q_i with $0 \leq q_1 \leq q_2 \leq ... \leq q_m$. Without loss of generality we can assume that $n = m$.[2]. For any $k \geq 1$ and for any vector $\gamma = (\gamma_1, \gamma_2, .., \gamma_k)$ we denote by $\gamma_{(j)}$ the jth smallest coordinate of γ, so that $\gamma_{(1)} \leq \gamma_{(2)} \leq ... \leq \gamma_{(k)}$.

The agents can be served only upon arrival. After an item is assigned, it cannot be reallocated. Each agent is characterized by a "type" x_j, and agents' types are governed by IID random variables $X_j = X$ on $[0, +\infty)$ with common cdf. F. Types are observable to the designer upon the agents' respective arrival (thus, at each point in time there is uncertainty about the types of agents arriving in the future). If an item with type $q_i \geq 0$ is assigned to an agent with type x_j, this agent enjoys a utility of $q_i x_j$. The designer wants to assign the items to the arriving agents in such a way as to maximize the expectation of the sum of agents' utilities.

Theorem 1 *Consider the arrival of an agent with type x_k in period $k \geq 1$. There exist $k+1$ constants $0 = a_{0,k} \leq a_{1,k} \leq ... \leq a_{k,k} = \infty$ such that:*

1. *The dynamically efficient policy assigns the item with quality $q_{(i)}$ - out of the remaining inventory - if $x_k \in (a_{i-1,k}, a_{i,k}]$.*

2. *In a problem with n periods the total expected welfare is given by*

$$W_n = \sum_{i=1}^{n} q_i a_{i,n+1}. \tag{1}$$

[1] The more recent literature is surveyed in Bergemann and Said [4] who differentiate among models where new agents or new information arrive over time. Of course, there were many antecedents, such as Dolan [12], Riley and Zeckhauser[29] or Wang [31] (arriving agents), and Courty and Li [7] (arriving information).

[2] If $m > n$, then only the n items with highest qualities are relevant for our study; if $m < n$, then we add $n - m$ dummy objects with quality $q = 0$.

Note that the cutoff a_{ik} that is being used at period k equals the expected value (in a problem with $k-1$ periods) of the type that gets the object with quality $q_{(i)}$, assuming that an optimal policy will be used. The remarkable part of the above result, due to Derman, Lieberman and Ross [9] (DLR hereafter), is that the precise values of the available qualities do not play any role for the optimal policy.

It order to better understand the main change induced by sequentiality, it is interesting to compare the above characterization to the one in the static case, where all agents arrive at once. Then we have a classic matching problem a la Becker [3], and the assignment is assortative. Thus, the expected welfare in the static case is given by

$$W_n^s = \sum_{i=1}^{n} q_i \mu_{i,n}. \tag{2}$$

where $\mu_{i,n}$ denotes the expectation of the *ith* lowest order statistic out of n copies of X. For the comparison of the static and dynamic scenarios we need the following definition:

Definition 2 *We say that vector $\alpha = (\alpha_1, \alpha_2, .., \alpha_n)$ is* majorized *by vector $\beta = (\beta_1, \beta_2, .., \beta_n)$ and we write $\alpha \prec \beta$ if the following system of $n-1$ inequalities and one equality is satisfied:*

$$\begin{aligned}
\alpha_{(k)} &\leq \beta_{(k)} \\
\alpha_{(k)} + \alpha_{(k-1)} &\leq \beta_{(k)} + \beta_{(k-1)} \\
... &\leq ... \\
\alpha_{(k)} + \alpha_{(k-1)} + ... + \alpha_{(2)} &\leq \beta_{(k)} + \beta_{(k-1)} + ... + \beta_{(2)} \\
\alpha_{(k)} + \alpha_{(k-1)} + ... + \alpha_{(n)} &= \beta_{(k)} + \beta_{(k-1)} + ... + \beta_{(n)}
\end{aligned}$$

Theorem 3 *For each n , the vector $\{a_{i,n+1}\}_{i=1}^{n}$ is majorized by the vector $\{\mu_{i,n}\}_{i=1}^{n}$.*

Proof. Assume first that all available qualities are equal, say $q_i = 1$, for all i. Then it is obvious that the sequential arrivals do not impose any loss of welfare. Thus, formulas (1) and (2) yield the one equality needed in the definition of majorization. In order to prove that all required inequalities hold note that expected welfare in the dynamic case - where assignment "mistakes" occur because of the sequential arrival - can never be higher than the welfare attained in the static case. The result follows by applying formulas (1) and (2) to the $n-1$ vector of qualities of the form $(1, 0, .., 0), (1, 1, 0, .., 0), ..., (1, 1, .., 1, 0)$. ∎

If $\alpha \prec \beta$,the vector α is less "variable" than the vector β, and majorization is the discrete version of second-order stochastic dominance. The above result makes precise the sense in which allocation "mistakes" due to the sequential nature of the allocation process yield a more "compressed" optimal policy and a reduced expected welfare.

3 Vickrey vs. Derman-Lieberman-Ross

Assume now that agents' types are private information, and that monetary transfers are possible. If an agent with type x obtains an item with quality q for price p then his utility is given by $qx - p$. In a deterministic, direct mechanism, every agent reports upon arrival his characteristic x_i and the mechanism specifies:[3] **1**. A non-random allocation rule (which object is allocated, if any) that only depends on arrival period, on the declared type of the arriving agent, and on the inventory of items available at that period; **2.** A payment to be made by the arriving agent which depends on the arrival period, on the declared type of the agent, and on the inventory of items available at that period.

A general mechanism design approach starts by a characterization of all dynamically implementable allocation policies. Roughly speaking, such policies are characterized by a monotonicity property: at each point in time, an agent with a higher type obtains an item with a higher quality (see Gershkov and Moldovanu [20]). In particular, in deterministic mechanisms, the set of types that obtains a given quality is an interval. It is immediate that the dynamically efficient policy identified by DLR has the required property.

Theorem 4 *The complete information, dynamically optimal policy is implementable also under incomplete information. The necessary payments form the dynamic analogue of the Vickrey-Clarke-Groves transfers, and are given by*

$$\begin{aligned} P_k\left(x,\Pi_k\right) &= \sum_{i=1}^{j}(q_{(i:\Pi_k)} - q_{(i-1:\Pi_k)})a_{i-1,k} \quad \text{if} \quad x \in [a_{j-1,k}, a_{j,k}), \qquad (3) \\ j &= 1,2,..k. \end{aligned}$$

where Π_k *is the inventory available at stage* k *and* $q_{(i:\Pi_k)}$ *is the ith lowest quality in* Π_k.

Payments have here an intuitive interpretation: look at static case with k objects and k agents where, in addition to the arriving agent with type x, there are $k-1$ "dummies" with types $a_{1,k} \leq ... \leq a_{k-1,k}$. These types are proxies for future agents. Then, if $x \in [a_{j-1,k}, a_{j,k})$, the arriving agent gets the object with quality $q_{(j:\Pi_k)}$, and the above payment represents the (negative) externality imposed on the dummy agents. This is, precisely, the dynamic extension of the celebrated Vickrey-Clarke-Groves idea.[4].

The main thing to note is that the above prices have a rather complex structure that does depend on the available qualities at each point in time. The mechanism design approach - with its focus on implementable allocations rather than on prices - yielded first the simply structured welfare maximizing

[3] There is no loss of generality here when we restrict attention to such mechanisms.

[4] Dolan [12] pioneered the use of dynamic versions of the Vickrey-Clarke-Groves mechanisms in order to achieve efficient queue disciplines. Recent general extensions of VCG schemes to dynamic settings can be found in Athey and Segal [2], Bergemann and Valimäki [5], and Parkes and Singh [28].

allocation. The associated prices follow then, basically, by a standard payoff equivalence exercise (see Myerson [24]). Analogously, it is much easier to find here, say, the revenue maximizing allocation and then derive associated prices rather than trying to directly derive revenue maximizing prices.

We conclude this brief illustration by describing several extensions of the above model/results.

4 Stochastic Arrivals

The generalization of the DLR model to continuous time and stochastic arrivals is due to Albright [1]. As above, implementable policies are characterized by cutoffs and satisfy a monotonicity property. The efficient allocation is implementable via a dynamic Vickrey-Clarke-Groves mechanism (see Gershkov and Moldovanu[18]). Budget- balancedness can be attained in the limit where the deadline and/or the arrival rate go to infinity.

5 Revenue Maximization

One of the classical contributions in RM is by Gallego and van Ryzin [14]. In their model a set of identical units is allocated until a deadline to unit-demand, stochastically arriving buyers with privately known valuations (Poisson arrivals)[5]. Under complete information, this model is a special case of Albright [1], who allows for several **heterogenous** objects. In Gershkov and Moldovanu [16] we add incomplete information to Albright's model, and first compute the revenue generated by any individual-rational, deterministic, Markovian and implementable policy[6]. The revenue maximizing policy - characterized by a variational argument - is, at each point in time, and for each subset of available objects, consisting of several cut-offs which determine the object allocated to the arriving agent. The associated optimal prices are, as in the above Section, completely determined by the implementation conditions. Interestingly enough, the fact that the cutoffs in the optimal policy do not depend on the precise values of the available qualities allows us to actually deduce them from the classical study with **identical** objects, due to Gallego and van Ryzin [14] ![7]

Gallien [15], Board and Skrzypacz [6], Gershkov and Moldovanu [19] consider buyers who are long-lived (and thus strategic with respect to their purchase times), while Mierendorff [23] and Pai and Vohra [27] focus on agents who are strategic with respect to the deadline by which they need the objects. The buyers' strategic behavior also plays a main role in Dana [10], Gale and Holmes [13] and Nocke, Peitz and Rosar [26] who have focused on the optimality of

[5] A generalization to multi-unit demand buyers - this is a dynamic and stochastic knapsack problem - appears in Dizdar et al. [11].

[6] The focus on such policies is without loss of generality.

[7] We also obtain a set of intuitive equations that characterize the optimal number of objects and their respective qualities. Finally, we explain why the average clearance mark-down is higher for the higher quality product lines, as empirically observed in a variety of settings.

advanced purchase discounts in frameworks where uncertainty is resolved over time.

6 Learning

Although very rare in the mechanism design literature (with its static focus), the assumption of gradual learning about an unknown environment is descriptive of most real-life dynamic allocation problems. But all above mentioned models assumed away such features. When learning about the environment takes place, the information revealed by a strategic agent affects both the current and the option values attached by the designer to various allocations. Since option values for the future serve as proxies for the values of allocating resources to other (future) agents, the private values model with learning indirectly generates informational externalities.[8] The efficient dynamic allocation in the DLR model with learning, say, need not be implementable (see Gershkov and Moldovanu [17]). In Gershkov and Moldovanu [20] we characterize the incentive-compatible dynamic policy that maximizes expected welfare while respecting incentive compatibility (second best). In particular, we show that this policy is deterministic, and that it satisfies a generalized form of a reservation price property appearing in classical search models. We also offer sufficient conditions under which the second-best policy coincides with the first-best. Roughly speaking, the main requirement puts a bound on the allowed optimism associated to higher observations in each period of search.

7 Conclusion

Many of the questions addressed in the Revenue/Yield Management literature are amenable to an analysis that uses the tools of Mechanism Design. A modern theory of RM should combine the dynamic models from Operations Research, Management Science and Computer Science on the one hand, and the powerful tools of Mechanism Design on the other. Such a combination would allow the study of many more interesting strategic situations that appear in real life applications.

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