# Bayesian and Dominant Strategy Implementation Revisited

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## Abstract

We study the equivalence between Bayes-Nash Incentive Compatibility (BIC) and Dominant Strategy Incentive Compatibility (DIC) in a standard social choice environment with linear utility and one-dimensional types. We consider two notions of equivalence: one based on agents' interim utilities (*U-equivalence*), and a stronger one based on conditional expected probabilities of choosing each alternative (*P-equivalence*). Our main results are: 1) For any BIC mechanism there is a P-equivalent DIC mechanism in settings with two social alternatives. 2) We construct a symmetric, BIC mechanism for which there is no P-equivalent DIC mechanism in an example with three alternatives. 3) In general symmetric settings, for any symmetric BIC mechanism there exists a symmetric, DIC and U-equivalent mechanism.

Our insights are based on elegant mathematical results about the existence of monotone measures with given monotone marginals.

# 1    Introduction

In an important and surprising contribution Manelli and Vincent [2010] focus on the standard one-object auction setting with private values and independent types, and show that for any Bayes-Nash Incentive Compatible (BIC) mechanism there exists a Dominant-Strategy Incentive Compatible (DIC) mechanism that yields, for each bidder and each type of this bidder, the same conditional expected probability of obtaining the object as in the original mechanism and hence, by payoff equivalence, the same interim expected utility.

In this paper we look at a standard, general social choice environment with linear utility and one-dimensional types which includes as a special case the one-object auction considered by Manelli and Vincent. We consider two notions of equivalence between mechanisms[1]: equivalence based on agents' interim utilities (*U-equivalence*), and equivalence based on conditional expected probabilities of choosing each alternative (*P-equivalence*). These two notions coincide in the Manelli and Vincent setting (where U-equivalence necessarily implies equivalence in conditional expected probabilities of obtaining the object for each bidder), or in any social choice setting with two alternatives. But, in general, P-equivalence is stronger than U-equivalence in settings when there are more than two alternatives. On the other hand, P-equivalence is weaker than an earlier notion used by Mookherjee and Reichelstein [1992] who require that two equivalent mechanisms provide the same ex-post probabilities for each alternative. Note that the primitive object in mechanism design is the social choice function, consisting here of a physical allocation and monetary transfers as functions of private information. Utilities are separate objects. Accordingly, most of the literature has been indeed concerned with properties of social choice functions (or correspondences). While P-equivalence follows this tradition, U-equivalence blends social choice functions and utilities in a specific way.

Our main results are: 1) For any settings with two alternative we show that, for any BIC mechanism there is a P-equivalent (and hence U-equivalent) DIC mechanism. 2) For any symmetric setting we show that for any symmetric BIC mechanism there exists a symmetric, DIC and U-equivalent mechanism. 3) For an example with three alternative we construct a symmetric,

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[1]We are extremely grateful to an anonymous referee for encouraging us to explicitly address the two notions.

BIC mechanism for which there is no P-equivalent DIC mechanism.

Our entire analysis is based on elegant mathematical results and ideas that, somewhat surprisingly, have been ignored so far in the Economics/Mechanism Design literature[2]. The main role is played by a result due to Gutmann et al. [1991]: For any bounded, non-negative function of several variables that generates monotone, one dimensional marginals, it is possible to generate the same marginals (i.e., the same expectations over all variables but one) with another non-negative function that is monotone in each coordinate, and that respects the same bound[3]. It is important to note that the difficulty in this result stems from the constraint of keeping the same bound.

The connection to BIC-DIC equivalence should be now obvious since in the independent private values model with quasi-linear utility and monetary transfers, DIC mechanisms are characterized by monotone allocations, which are described by probabilities of choosing various alternatives, while BIC mechanisms are characterized by monotone conditional expected allocations, which are obtained as marginals of the actual allocation.

Why do we need several notions of equivalence? A utilitarian designer who only takes into account the actual players will be indifferent between two U-equivalent mechanisms. But it is obvious that non-utilitarian designers - that may have personal preferences of their own, or that may take into account preferences of agents not currently in the game - need not be indifferent between two U-equivalent mechanisms. Here are two such examples:

1) Consider a risk averse/risk loving designer that cares about the agents' expected utilities and that has additional costs/benefits of implementing various alternatives. U-equivalence does not necessarily preserve ex-ante probabilities of choosing the alternatives (see for example our U-equivalent construction in the proof of Proposition 2, and the Remark after it). Hence such a designer need not be indifferent among two U-equivalent mechanisms, while she is necessarily indifferent among two P-equivalent mechanisms - they produce, by definition, the same ex-ante probabilities of choosing the various social alternatives.

2) Consider a dynamic setting where the designer takes a public decision

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[2]Building upon our work, Goeree and Kushnir [2011] provide a short proof of Manelli and Vincent's equivalence result by generalizing the proof of the Gutmann et al. [1991] theorem.

[3]Gutmann et al. build upon earlier contributions due to Lorenz [1949], Kellerer [1961] and Strassen [1965], that study the existence of measures with given marginals. These studies are in fact relevant for the analysis of reduced form auctions, e.g., Border[1991].

that affects both current and future agents. While the distribution of the values of the current agents may be known, the distribution of values for future agents may be yet unknown, and may depend on current realizations. Thus, from the point of view of the designer, the private information of current agents naturally enters the proxy utility functions for future agents, so even a utilitarian social planner will not be indifferent between U-equivalent mechanisms for the current agents.

Stronger notions of equivalence ensure that a richer set of designers is indifferent among two mechanisms that satisfy the respective requirements. Our counter-example shows that there is a price to be paid: even the transition from U- to the mildly stronger P-equivalence implies that, for non-utilitarian designers, implementation in dominant strategies offers strictly less freedom relative to Bayesian implementation.

## 2 Model and Preliminaries

There are $K$ social alternatives and $N$ agents. The utility of agent $i$ in alternative $k$ is given by $a_i^k x_i + c_i^k + t_i$ where $x_i \in [0, 1]$ is agent $i$'s private type, where $a_i^k, c_i^k \in \mathbb{R}$ with $a_i^k \geq 0$, and where $t_i \in \mathbb{R}$ is a monetary transfer. Types are drawn independently of each other, according to strictly increasing distributions $F_i$. Type $x_i$ is private information of agent $i$.

Note that the one-object auction analyzed in Manelli and Vincent [2010] is the special case of the above model where $K = N$, where $a_i^i = 1$, $a_i^j = 0$ for any $j \neq i$, and where $c_i^k = 0$ for any $i, k$.

A direct revelation mechanism $\mathbf{M}$ is given by $K$ functions $q^k : [0, 1]^N \to [0, 1]$ and $N$ functions $t_i : [0, 1]^N \to \mathbb{R}$ where $q^k(x_1, ..., x_N)$ is the probability with which alternative $k$ is chosen, and $t_i(x_1, ..., x_N)$ is the transfer to agent $i$ if the agents report types $x_1, ..., x_N$. Note that $\sum_{k=1}^{K} q^k(x_1, ..., x_N) = 1$ for each vector of reports $\mathbf{x} = (x_1, ..., x_N) \in [0, 1]^N$.

A direct revelation mechanism $\mathbf{M}$ is *Dominant-Strategy Incentive Compatible* (DIC) if truth-telling constitutes a dominant strategy equilibrium in the game defined by $\mathbf{M}$ and the given utility functions. A direct revelation mechanism $\mathbf{M}$ is *Bayes-Nash Incentive Compatible* (BIC) if truth-telling constitutes a Bayes-Nash equilibrium in the game defined by $\mathbf{M}$ and the given utility functions. Obviously, a DIC mechanism is a fortiori BIC.

Given mechanism $\mathbf{M}$, define for each $i, k$, one-dimensional marginal with

respect to coordinate $i$ as

$$Q_i^k(\widehat{x}_i) = \int_{[0,1]^{N-1}} q^k(x_1, ..., x_i, \widehat{x}_i, x_{i+1}, ..., x_N) dF_{-i}$$

where $dF_{-i} = dF_1...dF_{i-1}dF_{i+1}...dF_N$. This expression represents the conditional expected probability that alternative $k$ is chosen if all agents $j \neq i$ report truthfully while agent $i$ reports type $\widehat{x}_i$.

A necessary condition for $\mathbf{M}$ to be BIC is that, for each agent $i$, the function $\sum_{k=1}^K a_i^k Q_i^k(x_i)$ is non-decreasing. Moreover, a standard argument that follows from the incentive compatibility constraint implies that any $K$ functions $q^k$ that satisfy this condition are part of a BIC mechanism.

Analogously, a necessary condition for $\mathbf{M}$ to be DIC is that, for each agent $i$, and for any signals of the other agents, the function $\sum_{k=1}^K a_i^k q^k(x_1, ..., x_N)$ is non-decreasing in $x_i$. Any $K$ functions $q^k$ that satisfy this condition are part of a DIC mechanism.

**Definition 1**    *1. Two mechanisms $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ are P-equivalent if, for each $i, k$ and $x_i$, it holds that $Q_i^k(x_i) = \widetilde{Q}_i^k(x_i)$, where $Q_i^k$ and $\widetilde{Q}_i^k$ are the conditional expected probabilities associated with $\mathbf{M}$ and $\widetilde{\mathbf{M}}$, respectively.*

*2. Two mechanisms $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ are U-equivalent if they provide the same interim utilities for each agent $i$ and each type $x_i$ of agent $i$.*

Note that, for each agent $i$, interim utility is obtained (up to a constant) by integrating the function $\sum_{k=1}^K a_i^k Q_i^k(x_i)$ with respect to $x_i$ - this is a consequence of payoff equivalence. Thus $P$-equivalence implies $U$-equivalence.

The main tool in the subsequent analysis is the following result, which is a simple consequence of an elegant result due to Gutmann et al. [1991][4]:

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[4]Gutmann et al. formulate their result, Theorem 7, in terms of non-decreasing marginals only. Our extension is based on an immediate re-arrangement argument. Moreover, they only consider marginals with respect to the Lebesgue measure. A simple argument can be used to extend it to marginals with respect to product measures of the form $dF_{-i} = dF_1...dF_{i-1}dF_{i+1}...dF_N$ as needed in our application. This is done by considering the change of variables $u_i = F(x_i)$.

**Theorem 1** *Consider an integrable function $0 \leq q \leq 1$ on $\mathbb{R}^N$ having non-decreasing one-dimensional marginals with respect to coordinates $i \in D \subseteq [1, 2, ..., N]$, and non-increasing one-dimensional marginals with respect to coordinates $j \in D^C = [1, 2, ..., N] \setminus D$. Then there exists a function $0 \leq \psi \leq 1$, with exactly the same marginals, such that $\psi$ is non-decreasing in each coordinate $i \in D$, and is non-increasing in each coordinate $j \in D^C$.*

## 3  BIC-DIC Equivalence for Two Alternatives

In this section we consider settings with two social alternatives only. Since $q^2(x_1, ..., x_N) = 1 - q^1(x_1, ..., x_N)$, we obtain that $\sum_{k=1}^{2} a_i^k Q_i^k(x_i) = a_i^2 + (a_i^1 - a_i^2)Q_i^1(x_i)$, and therefore $U$-equivalence implies $P$-equivalence, and the two notions coincide here. In order to avoid trivial cases, we also assume that $a_i^1 \neq a_i^2$, for all agents $i$.

**Proposition 1** *Assume that $K = 2$. Then for any BIC mechanism there exists a $P$-equivalent (and thus $U$-equivalent) DIC mechanism.*

**Proof.** Since $K = 2$ and $q^2(x_1, ..., x_n) = 1 - q^1(x_1, ..., x_n)$, the allocation function in any mechanism can be represented by just one function $q^1(x_1, ..., x_n)$, the probability that alternative 1 is chosen. For BIC mechanisms we obtain for each $i$ that the function $\sum_{k=1}^{K} a_i^k Q_i^k(x_i) = a_i^2 + (a_i^1 - a_i^2)Q_i^1(x_i)$ is non-decreasing. In particular, $Q_i^1$ is non-decreasing if $a_i^1 - a_i^2 > 0$ and $Q_i^1$ is non-increasing if $a_i^1 - a_i^2 < 0$. Thus, the function $q^1(x_1, ..., x_n)$ satisfies the conditions in the Theorem 1 with $D = \{i \mid a_i^1 - a_i^2 > 0\}$. We obtain another function $0 \leq \psi \leq 1$, with exactly the same marginals, such that $\psi$ is non-decreasing in each coordinate $i \in D$, and non-increasing in each coordinate $j \in D^C$. As a consequence, $a_i^1 \psi(x_1, ..., x_n) + a_i^2[1 - \psi(x_1, ..., x_n)] = a_i^2 + (a_i^1 - a_i^2)\psi(x_1, ..., x_n)$ is non-decreasing in $x_i$ for any $i$. Together with appropriate transfers, $\psi(x_1, ..., x_n)$ defines a DIC mechanism which is $P$-equivalent to the given BIC mechanism. ∎

## 4  BIC-DIC Equivalence for Three or More Alternatives

In this section, we demonstrate that $P$-equivalence and $U$-equivalence differ for settings with three or more alternatives. We first construct a (symmetric)

BIC mechanism that has no $P$-equivalent DIC mechanism. This counterexample illustrates that $P$-equivalence between BIC and DIC fails in general when there are three or more alternatives. In this symmetric example we also note that $U-$equivalence holds. This observation can be then easily generalized to show that for any BIC mechanism there exists a $U$-equivalent DIC mechanism in the general symmetric environment (allowing for any number of agents and alternatives).

## 4.1   A Counterexample for Three Alternatives

We now construct a BIC mechanism that has no $P$-equivalent DIC mechanism.

The key observation behind the construction is as follows: In the Manelli-Vincent auction model and in settings with two alternatives, the BIC monotonicity condition can be separately imposed on the conditional probabilities $Q_i^k(x_i)$. But, in our general model the BIC monotonicity condition for agent $i$ is imposed on the function $\sum_{k=1}^{K} a_i^k Q_i^k(x_i)$. While $U$-equivalence requires that, for each agent $i$, the aggregated function $\sum_{k=1}^{K} a_i^k Q_i^k(x_i)$ is kept fixed, $P$-equivalence fixes separately each conditional expected probability of choosing alternative $k$, $Q_i^k(x_i)$ - this is mathematically more demanding and cannot always be attained.

We consider a setting with 2 agents, and 3 alternatives called $A, B, V$. It will be clear that the impossibility of $P-$equivalence is not a knife-edge phenomenon.

Agent 1 has types $x_1 > x_2$, agent 2 has types $y_1 > y_2$. Types are drawn independently from the uniform distribution, e.g. each type is drawn with probability $1/2$.[5]

The utility of agent 1 with type $x_i$ $(i = 1, 2)$, exclusive of transfers, is given by: $ax_i + c$ in alternative $A$; $x_i + d$ in alternative $B$; $v$ in $V$. The utility for agent 2 is obtained by plugging $y_i$ instead $x_i$ in these expressions. We further assume that $0 < a < 1$.[6]

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[5]This discrete setting allows us to clearly illustrate the difficulty in the construction. The example can be extend to continuous distributions that, say, put almost all mass around two types. Note that a discrete setting allows even more flexibility when choosing transfers that complement a given monotonic allocation to form a DIC mechanism.

[6]A counter-example can be easily constructed also for asymmetric situations. But the main idea behind the construction is clearer in the present symmetric setting because less functions and parameters are involved.

An allocation is given by probabilities $\{q^A(x_i, y_j), q^B(x_i, y_j)\}_{1 \le i,j \le 2}$. Note that $q^A(x_i, y_j) + q^B(x_i, y_j) \le 1$, where $q^V(x_i, y_j) = 1 - q^A(x_i, y_j) - q^B(x_i, y_j)$ represents the probability of choosing alternative $V$. In view of our previous result we need to ensure that $q^V$ is not identically zero.

The construction is divided in several steps.

**Step 1**: **Equivalence and Symmetry.**

Consider an (agent) symmetric allocation function $\{\widehat{q}^A(x_i, y_j), \widehat{q}^B(x_i, y_j)\}_{1 \le i,j \le 2}$, i.e., an allocation where $\widehat{q}^A(x_1, y_2) = \widehat{q}^A(x_2, y_1)$ and $\widehat{q}^B(x_1, y_2) = \widehat{q}^B(x_2, y_1)$. We first show that any $P-$equivalent allocation – that keeps all marginals fixed – has to be symmetric as well.

To see this, consider the conditional expected probabilities of choosing each alternative obtained from a given symmetric allocation. We have:

$$
\begin{aligned}
Q_1^A(x_1) &= \frac{1}{2}[\widehat{q}^A(x_1, y_1) + \widehat{q}^A(x_1, y_2)] \\
&= \frac{1}{2}[\widehat{q}^A(x_1, y_1) + \widehat{q}^A(x_2, y_1)] = Q_2^A(y_1) \\
Q_1^B(x_1) &= \frac{1}{2}[\widehat{q}^B(x_1, y_1) + \widehat{q}^B(x_1, y_2)] \\
&= \frac{1}{2}[\widehat{q}^B(x_1, y_1) + \widehat{q}^B(x_2, y_1)] = Q_2^B(y_1)
\end{aligned}
$$

Consider any other equivalent allocation rule $\{\widetilde{q}^A(x_i, y_j), \widetilde{q}^B(x_i, y_j)\}_{1 \le i,j \le 2}$. By equivalence we must have

$$
\begin{aligned}
Q_1^A(x_1) &= \frac{1}{2}\widetilde{q}^A(x_1, y_1) + \frac{1}{2}\widetilde{q}^A(x_1, y_2) \\
Q_2^A(y_1) &= \frac{1}{2}\widetilde{q}^A(x_1, y_1) + \frac{1}{2}\widetilde{q}^A(x_2, y_1)
\end{aligned}
$$

Together with $Q_1^A(x_1) = Q_2^A(y_1)$, this yields $\widetilde{q}^A(x_1, y_2) = \widetilde{q}^A(x_2, y_1)$. An analogous argument yields $\widetilde{q}^B(x_2, y_1) = \widetilde{q}^B(x_1, y_2)$. Thus $\{\widetilde{q}^A(x_i, y_j), \widetilde{q}^B(x_i, y_j)\}_{1 \le i,j \le 2}$ must be symmetric as well.

**Step 2: Construction of a symmetric BIC mechanism.**

We now construct a specific symmetric BIC mechanism for the above setting. Let $s$ be a small positive number, say, $s = 1/15$, and consider the following symmetric allocation (i.e., with $\widehat{q}^A(x_1, y_2) = \widehat{q}^A(x_2, y_1)$, $\widehat{q}^B(x_1, y_2) =$

$\widehat{q}^B(x_2, y_1))$ and corresponding conditional expected probabilities:

$$\begin{array}{ll}
\widehat{q}^A(x_1, y_1) = 13s & \widehat{q}^B(x_1, y_1) = as \\
\widehat{q}^A(x_1, y_2) = s & \widehat{q}^B(x_1, y_2) = as \\
Q^A(x_1) = 7s & Q^B(x_1) = as
\end{array} \tag{1}$$

and

$$\begin{array}{ll}
\widehat{q}^A(x_2, y_1) = s & \widehat{q}^B(x_2, y_1) = as \\
\widehat{q}^A(x_2, y_2) = s & \widehat{q}^B(x_2, y_2) = 9as \\
Q^A(x_2) = s & Q^B(x_2) = 5as
\end{array} \tag{2}$$

Note that $\widehat{q}^A(x_i, y_j), \widehat{q}^B(x_i, y_j) \in (0, 1)$ and $\widehat{q}^A(x_i, y_j) + \widehat{q}^B(x_i, y_j) < 1$. This last inequality is precisely the degree of freedom gained by having more than 2 alternatives.

The function $a\widehat{q}^A + \widehat{q}^B$ is not increasing in each coordinate separately, and therefore the mechanism is not DIC. But, the constructed mechanism satisfies

$$aQ^A(x_1) + Q^B(x_1) = 8as > 6as = aQ^A(x_2) + Q^B(x_2)$$

and analogously for agent 2. This "monotonicity on average" property implies that we can construct appropriate transfers that, together with $\widehat{q}$, yield a symmetric BIC mechanism.

**Step 3: An equivalent DIC mechanism does not exist.**

Assume now the existence of a DIC mechanism that is $P$-equivalent to the BIC mechanism constructed in Step 2. This DIC mechanism needs to be symmetric, as explained in Step 1. Therefore, it consists of 6 non-negative numbers

$$\begin{array}{lll}
q^A(x_1, y_1), \; q^A(x_1, y_2) & = & q^A(x_2, y_1), \; q^A(x_2, y_2) \\
q^B(x_1, y_1), \; q^B(x_1, y_2) & = & q^B(x_2, y_1), \; q^B(x_2, y_2)
\end{array}$$

that satisfy the following system of equations:

$$\begin{array}{l}
\frac{1}{2}q^A(x_i, y_1) + \frac{1}{2}q^A(x_i, y_2) = Q^A(x_i), \; i = 1, 2 \\
\frac{1}{2}q^B(x_i, y_1) + \frac{1}{2}q^B(x_i, y_2) = Q^B(x_i), \; i = 1, 2
\end{array} \tag{3}$$

where we omitted the redundant equations that need to hold for agent 2 with type $y_i$, $i = 1, 2$.

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Let's define the function $\psi\left(x_i, y_j\right)$ as follows:

$$\psi(x_i, y_j) = aq^A(x_i, y_j) + q^B(x_i, y_j), \; i, j = 1, 2 \tag{4}$$

Because $q^A\left(x_i, y_j\right), q^B\left(x_i, y_j\right) \in [0, 1]$ and $q^A\left(x_i, y_j\right) + q^B\left(x_i, y_j\right) < 1$, we obtain that $\psi(x_i, y_j) \in [0, 1)$. By symmetry we have $\psi(x_1, y_2) = \psi(x_2, y_1)$. Since the underlying mechanism $\{q^A, q^B\}$ is assumed to be DIC, $\psi\left(x_i, y_j\right)$ must be non-decreasing in each coordinate:

$$\psi(x_1, y_1) \geq \psi(x_1, y_2) = \psi(x_2, y_1) \geq \psi(x_2, y_2). \tag{5}$$

It follows from (3) that

$$\begin{array}{l} \frac{1}{2}\psi(x_1, y_1) + \frac{1}{2}\psi(x_1, y_2) = aQ^A(x_1) + Q^B(x_1) = 8as \\ \frac{1}{2}\psi(x_1, y_2) + \frac{1}{2}\psi(x_2, y_2) = aQ^A(x_2) + Q^B(x_2) = 6as \end{array} \tag{6}$$

Together with inequalities (5), equations (6) yield the following necessary bounds:

$$\begin{array}{l} 8as \leq \psi(x_1, y_1) \leq 10as \\ 6as \leq \psi(x_1, y_2) = \psi(x_2, y_1) \leq 8as \\ 4as \leq \psi(x_2, y_2) \leq 6as \end{array} \tag{7}$$

Note that $\psi(x, y)$ is a solution to the (discrete) Gutmann et. al problem of finding a function, monotone in $x$ and $y$ separately, with given monotone marginals of the form $aQ^A(x_i) + Q^B(x_i)$, $aQ^A(y_i) + Q^B(y_i)$, $i = 1, 2$. In fact, any DIC mechanism (not necessarily $P-$equivalent to the BIC mechanism constructed at Step 2) with fixed marginals $aQ^A + Q^B$ yields such a solution. Since we know by the discrete version of Theorem 1 that a solution to this problem does exist[7], the construction of a counterexample must hinge on the additional constraints imposed by $P-$equivalence, i.e., by equations 3.

Fix then $q^A(x_1, y_2)$ and use equations (3) and (4) to write all other five $q$'s in terms of $q^A(x_1, y_2)$:

$$
\begin{array}{rcl}
q^A(x_1, y_1) & = & 2Q^A(x_1) - q^A(x_1, y_2) \hspace{3cm} (8) \\
q^B(x_1, y_2) & = & \psi(x_1, y_2) - aq^A(x_1, y_2) \hspace{2.3cm} (9) \\
q^B(x_1, y_1) & = & \psi(x_1, y_1) - aq^A(x_1, y_1) \\
& = & \psi(x_1, y_1) - 2aQ^A(x_1) + aq^A(x_1, y_2) \hspace{0.5cm} (10) \\
q^A(x_2, y_2) & = & 2Q^A(x_2) - q^A(x_1, y_2) \hspace{2.4cm} (11) \\
q^B(x_2, y_2) & = & \psi(x_2, y_2) - aq^A(x_2, y_2) \\
& = & \psi(x_2, y_2) - 2aQ^A(x_2) + aq^A(x_1, y_2) \hspace{0.5cm} (12)
\end{array}
$$

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[7]The discrete version is also found in Gutmann et al.[1991]. In fact, the continuous result is obtained as a limit of the discrete one.

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We now check whether we can choose $q^A(x_1, y_2) \geq 0$, so that all other $q$'s are also non-negative.

Equations (8), (9), and (11) yield the necessary condition:

$$q^A(x_1, y_2) \leq \min \left\{ 2Q^A(x_1), 2Q^A(x_2), \frac{1}{a}\psi(x_1, y_2) \right\}.$$

Equations (10) and (12) yield the necessary condition:

$$q^A(x_1, y_2) \geq \max \left\{ 2Q^A(x_1) - \frac{1}{a}\psi(x_1, y_1), 2Q^A(x_2) - \frac{1}{a}\psi(x_2, y_2) \right\}.$$

Therefore, putting these two conditions together yields the necessary condition:

$$\max \left\{ 2Q^A(x_1) - \frac{1}{a}\psi(x_1, y_1), 2Q^A(x_2) - \frac{1}{a}\psi(x_2, y_2) \right\}$$
$$\leq \quad \min \left\{ 2Q^A(x_1), 2Q^A(x_2), \frac{1}{a}\psi(x_1, y_2) \right\} \tag{13}$$

By using now the the relations $Q^A(x_1) = 7s$, $Q^A(x_2) = s$, $\psi(x_1, y_1) = 16as - \psi(x_1, y_2)$, $\psi(x_2, y_2) = 12as - \psi(x_1, y_2)$,[8] we can rewrite the necessary condition (13) as

$$\max \left\{ -2s + \frac{1}{a}\psi(x_1, y_2), \; -10s + \frac{1}{a}\psi(x_1, y_2) \right\} \leq \quad \min \left\{ 14s, \; 2s, \; \frac{1}{a}\psi(x_1, y_2) \right\} \Leftrightarrow$$

$$-2s + \frac{1}{a}\psi(x_1, y_2) \leq \quad \min \left\{ 2s, \; \frac{1}{a}\psi(x_1, y_2) \right\} \Leftrightarrow$$

$$\frac{1}{a}\psi(x_1, y_2) \leq \quad 2s + \min \left\{ 2s, \; \frac{1}{a}\psi(x_1, y_2) \right\}$$

Since $2s + \min \left\{ 2s, \; \frac{1}{a}\psi(x_1, y_2) \right\} \leq 4s$, we obtain that a necessary condition for the above construction to be valid, i.e., for keeping all $q$'s non-negative, is that $\psi(x_1, y_2) \leq 4as$. But this contradicts the requirement $\psi(x_1, y_2) \geq 6as$, which was obtained in inequalities (7) at Step 3. Thus, a $P$-equivalent DIC mechanism cannot be constructed here. It should be clear from the above that the inexistence is not a knife-edge phenomenon – there are several degrees of freedom here in the choice of parameters.

**Q.E.D.**

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[8] See equations (1), (2) and (6).

## 4.2 $U$-Equivalence for Symmetric Settings

As we saw at Step 3 in the construction above , it was possible to construct an $U$-equivalent DIC mechanism in the symmetric setting of the counterexample. A straightforward generalization of that insight yields $U$-equivalence in the symmetric version of our general model, with any number of agents and alternatives. In order to apply the Gutmann et al. result, we normalize here coefficients in a suitable way.

**Proposition 2** *Assume that $a_i^k = a_j^k = a^k$ for all $k, i, j$, and that $F_i = F$ for all $i$. Moreover, assume that $0 = a^1 \leq a^2 \leq \cdots \leq a^K = 1$. Then for any symmetric, BIC mechanism there exists an $U$-equivalent symmetric DIC mechanism.*

**Proof.** Since $q^k(x_1, ..., x_N) \in [0, 1]$, $\sum_{k=1}^{K} q^k(x_1, ..., x_N) = 1$ and $a^k \in [0, 1]$ we have $\sum_{k=1}^{K} a^k q^k(x_1, ..., x_N) \in [0, 1]$. Moreover, since the original mechanism is BIC, the function $\sum_{k=1}^{K} a^k q^k(x_1, ..., x_N)$ has non-decreasing marginals. Symmetry implies that all the $N$ marginals are the same and given by $\sum_{k=1}^{K} a^k Q^k(x_i)$. Thus, the function $\sum_{k=1}^{K} a^k q^k(x_1, ..., x_N)$ satisfies the conditions in Theorem 1. Therefore, there exists a unique function $0 \leq \psi(x_1, ..., x_N) \leq 1$, with exactly the same marginals, such that $\psi$ is non-decreasing in each coordinate. Summing the terms $\psi(x_{\pi(1)}, ..., x_{\pi(N)})$ for all permutations $\pi$ of $x_1, ..., x_N$ (which does not affect monotonicity), and then dividing the sum by $N!$ yields a symmetric function, non-decreasing in each coordinate, that generates the original marginals.

To complete the proof we need to show that $\psi$ defines a feasible mechanism, i.e., to find functions $\widehat{q}^1, ..., \widehat{q}^K$ such that $\sum_{k=1}^{K} a^k \widehat{q}^k(x_1, ..., x_n) = \psi(x_1, ..., x_n)$ , and such that $\widehat{q}^k \in [0, 1]$ for any $k \in \{1, ..., K\}$. This is easily done, for example by setting $\widehat{q}^K(x_1, ..., x_n) = \psi(x_1, ..., x_n)$ and $\widehat{q}^1(x_1, ..., x_n) = 1 - \psi(x_1, ..., x_n)$. ∎

The above proof shows that it is possible to replicate the interim expected utilities of all the agents while using only two alternatives, with highest and lowest slope, respectively. In other words, U-equivalence does not necessarily ensure that the ex-ante probabilities of different alternatives are preserved.

**Remark 1** *It is worth noting that Theorem 1 has no counterpart for higher-dimensional marginals (or projections). From this perspective it seems unlikely that $P-$ equivalence can hold in sufficiently interesting multi-dimensional*

*models. Moreover, in the multi-dimensional case not every monotone alloca-
tion can be augmented by transfers in order to create an incentive compatible
mechanism. Indeed, Jehiel, Moldovanu and Stacchetti [1999] analyzed a stan-
dard multi-dimensional, private values model, interpreted as an one-object
auction with externalities, and computed a Bayes-Nash equilibrium in a two-
bidder auction with a reserve price whose conditional expected probabilities
cannot be replicated in dominant strategies.*

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