Tournaments with Midterm Reviews*

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Abstract

In many tournaments investments are made over time. The question whether to conduct a review once at the end, or additionally at points midway through the tournament, is a strategic decision. If the latter course is chosen, then the designer must establish both a rule for aggregating the results of the different reviews and a rule for determining compensations.

We first study the case of a fixed, exogenously given prize and then extend the analysis to case where the prize is not fixed but may vary with the tournament's outcome.

It is shown that (1) it is always optimal to assign a higher weight to the final review; (2) this weight increases with the dominance of the first-stage effort in determining the final review's outcome. When the prize is not fixed, the optimal design generates an asymmetric tournament in the second stage that favors the winner of the midterm review.

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1 Introduction

Lazear and Rosen (1981) originally observed that in many circumstances it is optimal to set up compensation on the basis of rank order, and that certain puzzling features of markets can easily be explained in such terms. Subsequent economic literature adopted their model in assuming that tournaments are like "all-pay auctions" in which agents choose their effort levels simultaneously at the start, with a given fixed prize being allocated according to the resulting ranking. A very partial list of this literature includes the papers of Green and Stokey (1983), Dixit (1987), Krishna and Morgan (1998), Nalebuff and Stiglitz (1983), and Moldovanu and Sela (2001). In many tournaments, however, investments are made over time and whether a review is conducted only once at the end, as in most tournament models, or additionally at one or more points midway through, is a strategic decision of the mechanism designer. An important question that arises is how the results of the different reviews are optimally aggregated into a ranking and what is the optimal compensation function that maps these aggregate outcomes into prizes. These are the issues that we seek to address in this paper.

Midway reviews are a common phenomenon in tournaments. Students compete to be ranked in the top of their class, and the professor must choose whether to give a final exam only, or a final *and* a midterm exam. In the latter case, a rule for aggregating the results of the two exams into a ranking must also be determined. Similarly, employees exert effort so as to be promoted in organizational hierarchies. Periodical reviews are conducted and these reviews are then aggregated to determine who is to be promoted and by how much his salary is to increase.

Our aim is to understand how reviews and prizes provide incentives to exert effort, and not merely how they are used to select the ablest agent among asymmetric ones. To this end, we study a simple two-stage two-agent tournament in which symmetric agents choose their effort level in stage one, and again in stage two. A designer whose goal is to maximize the agents' total effort has to decide whether to conduct only a final review, or both a midterm and a final review. The review process is not perfect and can yield only an ordinal ranking that is positively, but only partially, correlated with the agents' efforts. While the effort invested at stage two affects only the final review, the effort invested at stage one may affect the outcomes of both reviews. Finally, we assume that the outcome of a midterm review, if any, is public knowledge.

We first study the case of a fixed prize to be allocated to one of the agents. The assumption of a fixed prize best fits situations similar to class-ranking tournaments among students. It is shown that conducting a midterm review has two opposite effects: it tends to increase the agents' effort level at stage one, but to decrease it at stage two. While we show that it is always strictly optimal to conduct a midterm review, we also demonstrate that this is true *only* when the results of both reviews are aggregated optimally. In particular, we show that treating the two reviews symmetrically may result in an equilibrium effort level inferior to the one obtained when only a final review is conducted. However, treating the two reviews lexicographically by focusing on the final review, and using the midterm review merely as a tie-breaking rule, strictly dominates the option of conducting a final review only. As our characterization will show, the optimal mechanism falls somewhere in between these two extreme mechanisms. In particular, we shall show that the more effective the first-stage effort is in determining the final review's outcome, the smaller is the weight that should be assigned to the midterm review in determining the agents' ranking.

While the assumption of a fixed prize, standard in the literature, is often appropriate, it is not always so. For example, when the tournament is among workers competing for a promotion and a raise, the size of the prize is not fixed, but varies according to the tournament's outcome. Thus, unlike the case of a fixed prize where the designer's goal is simply to maximize the agents' expected efforts, when the prize is costly, the designer must take into account the trade-off between incentives to exert effort on the one hand and the cost of the prize on the other hand. Our main result establishes that it is optimal to give different incentives to the agents at the second stage. In particular, at the second stage the marginal payment to the agent who ranked first in the midterm review is higher than the marginal payment to the other agent. While such an asymmetric treatment reduces the incentives at the second stage, the higher effort it induces at the first stage more than compensates for the loss. Similarly, we show that the amount allocated to prizes is highest when one agent comes out first in both reviews, and lowest when both reviews result in a tie.

Related Literature

Rosen (1986) , Gradstein and Konrad (1999) and Moldovanu and Sela (2006), among others, studied a different version of multi-round tournament called the *Elimination Tournament*. In the elimination tournament the agents are divided into groups and only the winner of each round proceeds to the next round, where he competes against the winners of other groups. The goal is to design an optimal structure of prizes at every round and an optimal assignment of contestants into groups. Note, however, that in all of these multi–stage tournaments the aggregation rule is exogenously given and in particular once an agent loses in one stage, his probability of winning the prize is reduced to zero.

Meyer (1991) considered a different version of a multi–stage tournament in which in every stage a manager can only observe whether one of the agents outperforms his opponent by some margin. The manager chooses these margins in order to gain information on the workers' abilities. In particular, it is shown that it improves the manager's information if the last period margin is chosen in favor of the current leader. Our paper, in contrast, uses the result of the midterm review as a tool to increase the efforts chosen by the participants.

A paper that is closer to ours is Aoyagi (2004), who studied a multi–stage two-agent tournament. However′ unlike the case studied here, Aoyagi assumes a fixed mechanism, such that equal weights are assigned to all midway reviews. In an environment in which first-stage effort is as effective as second-stage effort, relative to the final outcome, Aoyagi addressed the question of when it is optimal to reveal to participants information about the outcome of the midterm review. Ederer (2006) enriched Aoyagi's model with agents who have different abilities. Yildirim (2005) also studied a two–stage two–agent contest in which agents observe each other′s effort in stage one before investing in stage two. However, in his model there is only one review at the end. Yildirim analyzes the effect of the asymmetric abilities on the equilibrium strategies of the players.

Dubey and Haimanko (2003) studied the effect of aggregating the results of reviews on the incentives of the contestants. They assume that the principal samples a number of rounds and the winner is the agent who wins the most rounds (among sampled). They show that as the number of rounds goes to infinity the proportion of stages that are sampled goes to zero. This result is driven by sufficient differences in the contestants' quality.

The remainder of the paper is organized as follows. In the first four sections we study the model in which a fixed prize is allocated to one of the agents. Section 2 describes the basic setup. In Section 3 we study the equilibrium when the designer is restricted to conducting a final review only. The equilibrium when midterm and final reviews are conducted is analyzed in Section 4, and the optimal aggregation rule is then characterized in Section 5. In Section 6 we relax the assumption of a fixed prize and allow the designer to award different prizes as a function of the reviews' outcome.

2 Basic Setup

Two risk-neutral agents $i = 1, 2$ are asked to exert effort in two stages. Agent $i's$ effort in stage $t = 1, 2$ is denoted by $e_i^t \in [0, \infty)$ and is exerted at cost $c : [0, \infty) \to \mathbb{R}_+$. Effort e_i^t is agent i's private information and is not observed by either the other agent or the principal. The principal, however, whose goal it is to maximize the expected sum of effort $\sum_i \sum_t e_i^t$, can influence the agents' decision by conducting reviews and rewarding the agents in a way that reflects the reviews' results. Reviews can take place either after stage one (hereafter the "midterm review"), after stage two (the "final review"), or after both stages, and we assume throughout the paper that the reviews' results are public information.

For now we restrict our attention to the case where there is a fixed prize of size one that has to be allocated at the end of stage two. The prize might be promotion to a higher rank in a corporation, or it might be the utility of a student for being ranked first in his class. This assumption is relaxed in Section 6. Thus, if agent i whose effort levels in the two stages are e_i^1 and e_i^2, respectively, wins the award in probability p, his expected payoff is $p - \sum_t c(e_i^t)$.

The review process is imprecise and yields only a noisy ranking of agents' efforts. In particular, the outcome of a midterm review, if conducted, is determined by

$$\Gamma\left(e_1^1, e_2^1\right) = \left[f_1\left(e_1^1 - e_2^1\right),\ \ f_0\left(e_1^1 - e_2^1\right),\ \ f_2\left(e_1^1 - e_2^1\right)\right],$$

where $f_1(\cdot)$ is the probability that agent 1 is ranked first in the review. Similarly, $f_2(\cdot)$ is the probability that agent 2 is ranked first, and $f_0(\cdot) = 1 - f_1(\cdot) - f_2(\cdot)$ is the probability that the review is inconclusive and a tie is declared.[1]

Let $\tau_i = \delta e_i^1 + e_i^2$ denote the effective total effort, where $\delta \in [0,1]$ captures the fact that the effort in stage one is not as effective as the effort in stage two in determining the outcome of the final review, which is similarly determined by

$$\Gamma\left(e_1^1, e_2^1, e_1^2, e_2^2\right) = \left[f_1\left(\tau_1 - \tau_2\right),\ \ f_0\left(\tau_1 - \tau_2\right),\ \ f_2\left(\tau_1 - \tau_2\right)\right].$$

We make the following assumption on $\Gamma(\cdot)$ and $c(\cdot)$:

As: Symmetry: for all $y \in (-\infty, \infty)$,

$$f_0(y) = f_0(-y), \ \ and \ \ f_1(y) = f_2(-y).$$

Ai: Information content

a. $\frac{df_1(x)}{dx} > 0$; $\frac{df_2(x)}{dx} < 0$; $\frac{df_0(x)}{dx} \begin{cases} \geq 0 \ \ if \ \ x < 0 \\ = 0 \ \ if \ \ x = 0 \\ \leq 0 \ \ if \ \ x > 0 \end{cases}$,

b. $\lim_{x\to-\infty} f_1(x) = 0$ and $\lim_{x\to\infty} f_1(x) = 1$.

At: Γ is twice continuously differentiable.

Ac: $c(0) = 0$, $c'(0) = 0$ and for any $e \in [0, \infty)$, $c''(e) > \gamma(\Gamma) > 0$.[2]

[1] The main results still hold if instead we define

$$\Gamma\left(e_1^1, e_2^1\right) = \left[f_1\left(\frac{e_1^1}{e_2^1}\right),\ f_0\left(\frac{e_1^1}{e_2^1}\right),\ f_2\left(\frac{e_1^1}{e_2^1}\right)\right].$$

This specification, however, suffers from the problem of non-existence of symmetric equilibrium as the agents utilities are not convex at the point of the symmetric first order conditions, unless very specific form of Γ is adopted.

[2] The precise form of $\gamma(\Gamma)$ is defined in Appendix B.

Note that **As** expresses the symmetry between the two agents, while **Ai** captures the idea that the review process is informative. More precisely, the probability of coming out first (second) increases (decreases) in one's own effort, while the probability of a tie is maximized when both agents choose the same effort level. Assumptions **At** and **Ac** are mainly technical and much more than what is needed to assure that second-order conditions for optimum are met and that a symmetric equilibrium exists. In particular, the cost function $c(\cdot)$ must be convex enough that its second derivative is always above some constant γ that in turn depends on the shape of Γ.

The figure below illustrates the main features of $\Gamma(\cdot)$. In particular, note that $f_1(0) = f_2(0) < 1/2$ and that $f_0(\cdot)$ reaches its maximum at 0.

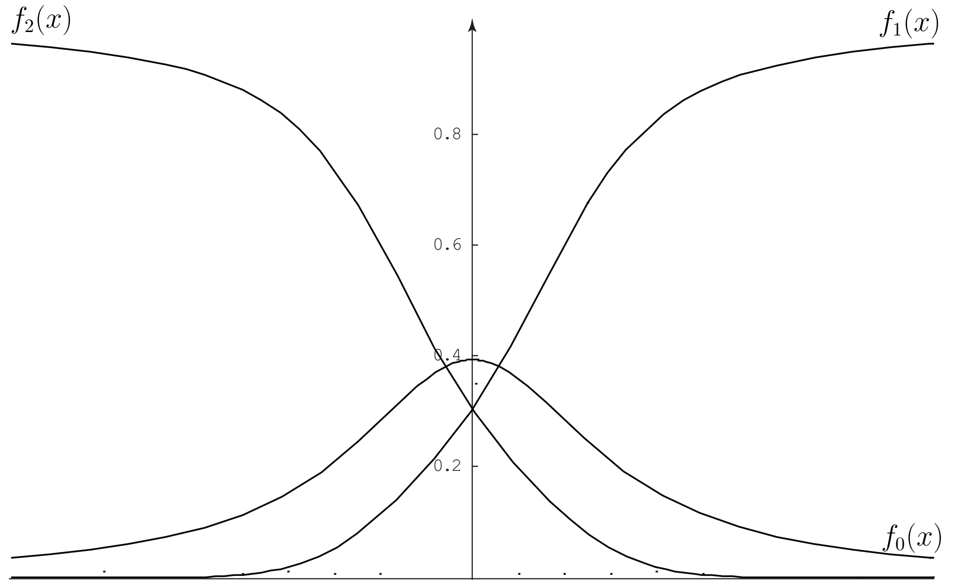

Figure 1: The main features of $\Gamma(\cdot)$.

Denote the output of $t's$ review by $s_t \in \{0, 1, 2\}$. A review system, often called a *mechanism*, specifies how many reviews to conduct, when to conduct them, and how the different s_t are then to be aggregated to yield an allocation rule. We restrict our attention to symmetric mechanisms.

Before we proceed, some words concerning the properties of $\Gamma\left(e_1^1, e_2^1, e_1^2, e_2^2\right)$ are in order. First note that $f(\cdot)$ is used to represent the review whether is a midterm review or a final. This captures the basic assumption that the noise in the model is coming solely from the review system. Specifically, we assume that an agent's output is his private information and is fully determined by his effort. Because outputs are unobservable, imprecise and coarse reviews are used as incentives to the agents to exert effort. For example, in the case of class-ranking tournaments among students, knowledge is determined by the time students put into studying, while exam grades, which are imperfectly correlated with knowledge, are used to provide incentives. Similarly, when a corporation conducts reviews among colleagues in a team to determine who is to be promoted, the noise is usually in the review process. This is in contrast to other models in the literature where it is assumed that outputs are *observable* but consist of effort and noise. Of course reviews are redundant when outputs are observable, but tournaments are not necessarily the optimal mechanism to use as observed outputs enable agents to be compensated according to the marginal product (see, however, Green and Stokey (1983)).[3]

In the current model, the precision of the review, captured by $f(\cdot)$, is exogenously fixed. A more realistic model might incorporate a costly choice of precision for $f(\cdot)$ and a trade-off between precision and cost as well as between the number of costly reviews and the precision of a given review. In such a model we might also see one precision for the midterm and a different one for the final. This is, however, beyond the scope of this paper.

We start by studying the mechanism where only one review is conducted.

3 Conducting Final Review Only

When only a final review is conducted, the set of symmetric mechanisms is characterized by the probability $\beta \in [0,1]$ at which the prize is allocated to the agent who was ranked first in the review. Thus, the expected utility of agent i, whose effort levels in the two stages are e_i^1 and e_i^2, and whose opponent's effort levels are e_j^1 and e_j^2, defined above as $\tau_l = \delta e_l^1 + e_l^2$ for $l = i, j$, is

$$-c(e_i^1) - c(e_i^2) + \beta f_i(\tau_1 - \tau_2) + \frac{1}{2} f_0(\tau_1 - \tau_2) + (1-\beta) f_j(\tau_1 - \tau_2) \tag{1}$$

which by **As** can be written as

$$-c(e_i^1) - c(e_i^2) + \beta f_1(\tau_i - \tau_j) + \frac{1}{2} f_0(\tau_i - \tau_j) + (1-\beta) f_2(\tau_i - \tau_j).$$

The two first-order conditions with respect to e_i^1 and e_i^2 are given by

$$c'(e_i^1) = \delta\left[\beta f_1'(\tau_i - \tau_j) + \frac{1}{2} f_0'(\tau_i - \tau_j) + (1-\beta) f_2'(\tau_i - \tau_j)\right] \tag{2}$$

and

$$c'(e_i^2) = \left[\beta f_1'(\tau_i - \tau_j) + \frac{1}{2} f_0'(\tau_i - \tau_j) + (1-\beta) f_2'(\tau_i - \tau_j)\right]. \tag{3}$$

First note that for $\beta \leq \frac{1}{2}$ the best response of each agent is to choose zero effort level in every stage. Also recall from Assumption **Ai** that $f_0'(0) = 0$. Therefore, for any $\beta > \frac{1}{2}$, there exists a

[3] We thank the referee for point out the difference between the current model and other models in the tournament literature.

symmetric solution to (2) and (3) where $e_1^1 = e_2^1 = \hat{e}^1$ and $e_1^2 = e_2^2 = \hat{e}^2$ for which

$$\begin{aligned} c'(\hat{e}^1) &= \delta(2\beta - 1) f_1'(0)\,, \\ c'(\hat{e}^2) &= (2\beta - 1) f_1'(0)\,. \end{aligned} \tag{4}$$

From the assumed convexity of $c(\cdot)$ and the monotonicity of f_1 it follows that a designer whose goal is to maximize the agents' efforts will set $\beta = 1$, which allows us to rewrite (4) as

$$\begin{aligned} c'(\hat{e}^1) &= \delta f_1'(0)\,, \\ c'(\hat{e}^2) &= f_1'(0)\,. \end{aligned} \tag{5}$$

In Appendix B we prove that second-order conditions for maximization are also satisfied. Thus, the solutions to (5) determines the effort level in the symmetric equilibrium.

Remark 1. *A second look at (4) reveals that the rule according to which the prize is allocated following a tie has no effect on incentives. In particular, choosing to allocate the prize with some probability $\alpha \in [0, 1)$ when the review is inconclusive will not change the effort level in equilibrium.* [4] *This is a consequence of Assumption (**Ai**) that a tie is most likely when both agents choose the same effort level. Thus, a small change in the effort level of any agent in either stage will not change the probability of a tie.*

Remark 2. *In our setup δ captures some elements that are under the principal's control and others that are not. For example, δ might capture the fact that studying for the final well ahead of time is less effective than cramming for the final, which of course is not under the principal's control. On the other hand, by shifting more of the final grade's weight to materials that are covered early on, the principal can make the time invested early on more effective. Similarly, in a promotion decision in a corporation, the committee in charge of promotion might be more affected by the latest achievements of the different candidates. In light of this discussion, it is instructive to note that, in equilibrium, the total effort level $\hat{e}^1 + \hat{e}^2$ increases with the discount factor δ.*

A corollary of this observation is that conducting only a final review dominates a mechanism in which only a midterm review is conducted. To see why, simply note that when only a midterm review is conducted, second-stage effort has no effect on the allocation of the prize, and hence the effort levels in equilibrium $\bar{e}^1$ and $\bar{e}^2$ are exactly the mirror image of the effort levels when only a final review is conducted and $\delta = 0$.

[4] In this case, the first-order conditions are

$$\begin{aligned} c'(e_i^1) &= \delta\left[\beta f_1'(\tau_i - \tau_j) + \frac{\alpha}{2} f_0'(\tau_i - \tau_j) + (1-\beta) f_2'(\tau_i - \tau_j)\right] \\ c'(e_i^2) &= \left[\beta f_1'(\tau_i - \tau_j) + \frac{\alpha}{2} f_0'(\tau_i - \tau_j) + (1-\beta) f_2'(\tau_i - \tau_j)\right] \end{aligned}$$

and their unique symmetric solution is still given by (4).

4 Conducting Midterm and Final Reviews

Recall that $s_t \in \{0,1,2\}$ stands for the result of review $t \in \{1,2\}$ and let $g(s_1,s_2)$ be the probability that the prize goes to agent 1 after the history (s_1,s_2). Because we are restricting our attention to symmetric mechanisms it follows that

$$g(s_1,s_2) = 1 - g(s_1', s_2') \text{ whenever } s_i' = \begin{cases} 0 & if \quad s_i = 0 \\ 2 & if \quad s_i = 1 \\ 1 & if \quad s_i = 2 \end{cases}.$$

Our interest lies in characterizing the optimal values of $g(s_1,s_2)$. At the end of stage one a midterm review is conducted and results either in a tie or in a winner.[5] Now, although the effort level of agent j in stage one is not revealed to agent i, in equilibrium agent i knows its value. Thus, by abusing the language somewhat, we refer to the different continuations following the midterm as subgames. Let $e_i^L(e_i^1)$ be agent i's optimal effort level in the subgame when he is the leader, after exerting e_i^1 in stage one. Similarly, let $u_i^L(e_i^1, e_i^L(e_i^1))$ denote his expected utility in the subgame and define $e_i^F(e_i^1),\ u_i^F(e_i^1, e_i^F(e_i^1))$ for the subgame in which he is a follower and $e_i^T(e_i^1)$, $u_1^T(e_1^1, e_1^T(e_1^1))$ for the subgame following a tie.

Agent $i's$ expected utility in the mechanism can now be written as

$$\begin{aligned} &-c(e_i^1) + f_i\left(e_1^1 - e_1^1\right) u_i^L(e_i^1, e_i^L(e_i^1)) + f_0\left(e_1^1 - e_1^1\right) u_i^T(e_i^1, e_i^T(e_i^1)) \\ &+ f_j\left(e_1^1 - e_1^1\right) u_i^F(e_i^1, e_i^F(e_i^1)). \end{aligned}$$

Using **As**, it can be rewritten as

$$\begin{aligned} &-c(e_i^1) + f_1\left(e_i^1 - e_j^1\right) u_i^L(e_i^1, e_i^L(e_i^1)) + f_0\left(e_i^1 - e_j^1\right) u_i^T(e_i^1, e_i^T(e_i^1)) \\ &+ f_2\left(e_i^1 - e_j^1\right) u_i^F(e_i^1, e_i^F(e_i^1)). \end{aligned}$$

In equilibrium, e_i^1, $e_i^L(e_i^1)$, $e_i^F(e_i^1)$, and $e_i^T(e_i^1)$ maximize agent i's payoff given the strategy of his rival, agent j, e_j^1, $e_j^L(e_j^1)$, $e_j^F(e_j^1)$, and $e_j^T(e_j^1)$. The following lemma, which derives from the First-Order Conditions of the agents' maximization problem in a symmetric equilibrium, is instrumental in characterizing the optimal mechanism. The proof of it is given in the appendix A.

[5] It is assumed that the result of the midterm review is revealed to both agents. In a more general model, the principal may decide also on the revelation policy. In particular, he may reveal to each player information about his own performance only. In our model, however, this type of revelation policy is equivalent to complete information revelation. For the characterization of optimal revelation policy we refer the reader to Aoyagi (2004) and Ederer (2006).

Lemma 1. *The effort levels $\overline{e}^1$, e^T, and $e^L = e^F = e^{LF}$ are a solution to First-Order Conditions if they satisfy the following three equations:*

$$c'(e^T) = \begin{cases} f_1'(0)\,(g(0,1) - g(0,2)) & \text{if} \quad g(0,1) - g(0,2) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$c'(e^{LF}) = \begin{cases} f_1'\,(0)\,(g(1,1) - g(1,2)) & \text{if} \quad g(1,1) - g(1,2) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

and

$$c'(\overline{e}^1) = \begin{cases} f_1'\,(0)\,A & \text{if} \quad A > 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where

$$\begin{aligned} A = {} & [2f_1\,(0)\,(g(1,1) + g(1,2) - 1) + f_0\,(0)\,(2g(1,0) - 1)] \\ & + 2f_1\,(0)\,\delta\,[g(1,1) - g(1,2)] + f_0\,(0)\,\delta\,[g(0,1) - g(0,2)]\,. \end{aligned}$$

Moreover, these effort levels are the unique symmetric solution to the First-Order Conditions.[6]

Proof. See Appendix A. □

Denote by $\overline{e}^1(g(s_1, s_2))$, $e^{LF}(g(s_1, s_2))$, and $e^T(g(s_1, s_2))$ the solutions to (6), (7), and (8).

Remark 3. *Note that the agents' efforts in period two are not affected by how the prize is allocated after a tie in the final review, as can be seen from the absence of the term $g(\cdot, 0)$ in either (7) or (6). But unlike the case where only final review is conducted, here $g(\cdot, 0)$ does have an effect on the agents' incentives, and in particular on the effort exerted in stage one, as can be seen in (8). Consequently, when two reviews are conducted, the allocation rule after a tie in the final must be chosen with care. However, the allocation rule after ties in both reviews $g(0, 0)$ has no effect on incentives (note that the term $g(0, 0)$ does not appear in (7), (6), or (8).)*

Remark 4. *One may wonder how it is that in a subgame where there is a leader, both agents, the leader and the follower, exert the same level of effort in equilibrium. The reason for this is rather simple: winning the subgame is of equal value to both leader and follower. For the follower it increases the probability of getting the prize by $g(2, 1) - g(2, 2)$, while for the leader it increases the probability by $g(1, 1) - g(1, 2)$. In any mechanism in which the prize is fixed, these probabilities must be the same. When the assumption that the prize is fixed is relaxed, this property does not hold; see Section 6.*

[6] The symmetry assumption corresponds here to the same efforts being made at the first and second stages after a tie.

5 The Optimal Allocation Rule

The optimal allocation rule $g^*(s_1, s_2)$ solves

$$\begin{aligned} g^*(s_1, s_2) &= \arg\max_{g(s_1,s_2)} [2\overline{e}^1(g(s_1, s_2)) + (1 - f_0(0))\, 2e^{LF}(g(s_1, s_2)) \\ &\quad + f_0(0)\, 2e^T(g(s_1, s_2))]. \end{aligned} \tag{9}$$

The following theorem follows immediately from the first-order conditions that were derived in Lemma 1, and the monotonicity of f_1 and c'.

Theorem 2. *In the optimal symmetric allocation rule $g^*(s_1, s_2)$,*[7]

$$\begin{aligned} g^*(1,1) &= 1 - g^*(2,2) = 1 \\ g^*(1,0) &= 1 - g^*(2,0) = 1 \\ g^*(0,1) &= 1 - g^*(0,2) = 1 \\ g^*(0,0) &= 0.5. \end{aligned}$$

Proof. From (8) and (7) it follows that effort levels in stage one and in the subgame in which there is a leader increase with $g^*(1,1)$. Because the effort level in the subgame in which there is a tie is not affected by $g^*(1,1)$, we conclude that in the optimal mechanism $g^*(1,1) = 1$. Similarly, $g^*(0,1) = 1$ follows because effort levels in stage one and in the subgame in which there is a tie increase with $g^*(0,1)$ (see (8) and (6)), while $g^*(0,1)$ does not have an effect on the effort level in the subgame in which there is a leader (see (7)). Finally, note that $g^*(1,0) = 1$ follows because the effort level in stage one increases with $g^*(1,0)$ (see (8)), but lt has no effect on the effort level in stage two. $g^*(0,0) = 0.5$ follows from the symmetry of the mechanism. □

In Appendix B we show that second-order conditions are satisfied. Thus the effort levels $\overline{e}^1(g^*(s_1, s_2))$, $e^{LF}(g^*(s_1, s_2))$, and $e^T(g^*(s_1, s_2))$ constitute a symmetric equilibrium. In other words, $\overline{e}^1(g^*(s_1, s_2))$, $e^{LF}(g^*(s_1, s_2))$, and $e^T(g^*(s_1, s_2))$ define a global maximum for one agent, given that the other agent is choosing the same levels of effort.

Theorem (2) characterizes all values of $g^*(s_1, s_2)$ except $g^*(1,2) = 1 - g^*(2,1)$, which turns out to depend on the specific parameters of the problem (like c and Γ) and will be studied shortly. But before doing so, and in light of Theorem (2), it is constructive to consider two commonly used aggregation rules, the *independent rule* and the *majority rule*.

[7] The allocation rule after two ties has no effect on incentives and hence can be chosen arbitrarily. The choice of $g^*(0,0) = 0.5$ follows from the symmetry of the mechanism.

- *The Independent Rule:* In the independent rule, each review is considered in isolation and the final allocation is then determined by some weighted average of the two results. That is,

  $$g^I(s_1, s_2) \equiv \alpha g_1(s_1) + (1 - \alpha) g_2(s_2)$$

  for some $\alpha \in [0, 1]$ where $g_i(s_i)$denotes the outcome of the i^{th} review. This rule is often used in academia when the midterm is typically given a lower weight than the final. One implication of Theorem (2) is that this rule is suboptimal. To see why, note first that according to this rule

  $$g^I(1, 1) - g^I(1, 0) = g^I(0, 1) - g^I(0, 0).$$

  However, it follows from Theorem (2) that in the optimal rule we must have

  $$0 = g^*(1, 1) - g^*(1, 0) \neq g^*(0, 1) - g^*(0, 0) = 0.5.$$

  The inferiority of the independent rule follows from the restriction to allocate a share α of the prize after the midterm regardless of its result. In particular, a share α is allocated even when the midterm results in a tie, which has no effect on the incentives. In this case, keeping this share of the prize available for allocation after the final will enhance incentives.

- *The Majority Rule:*In the majority rule, an agent is awarded (say) two points after winning a review, one point after a tie, and zero otherwise, and the prize is awarded to the agent who collects the most points. Note that all values of $g^*(s_1, s_2)$ that are determined by Theorem (2) agree with this simple and commonly used rule by which the two reviews are treated symmetrically.Though one might hasten to conclude that a natural candidate for $g^*(1, 2)$ is half, the value assigned to it by the majority rule, the following example demonstrates that choosing $g^*(1, 2) = 0.5$ is often suboptimal and might even lead to an outcome that is inferior to the one obtained when only a final review is conducted.

Example: *Assume that* $c(e) = \exp(e) - e - 1$, $\delta = 1$, *and*

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{(s - 0.2)^2}{2}) ds.$$

It follows that in this case

$$f_1'(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x - 0.2)^2}{2}\right)$$

and

$$\begin{aligned} f_1(0) &= 0.420\,74 \\ f_1'(0) &= 0.391\,04. \end{aligned}$$

Consider first the one-review system in which only a final review is conducted. In this mechanism, the first-order conditions yield $e_1 = e_2 = \bar{e}$ *where*

$$c'(\overline{e}) = f_1'(0).$$

It follows that in the one-review system, the effort level of each agent in each stage is $\bar{e} = 0.330\,05$.

Now consider the two-review system with the majority rule $g^m(s_1, s_2)$ *in which*

$$g^m(1,1) = g^m(1,0) = g^m(0,1) = 1 \quad \textit{and} \quad g^m(1,2) = g^m(0,0) = 0.5$$

is used. The solutions to the first-order conditions, $\bar{e}^1$, e^T, *and* e^{LF}, *are given by the system of equations*

$$\begin{aligned} c'(e^T) &= f_1'(0) \\ c'(e^{LF}) &= \frac{1}{2} f_1'(0) \\ c'(\overline{e}^1) &= (1+\delta) f_1'(0) \left(f_1(0) + f_0(0)\right), \end{aligned}$$

which yields

$$\overline{e}^1 = 0.373\,65, \quad e^{LF} = 0.178\,58 \ \textit{and} \ e^T = 0.330\,05.$$

Thus, the expected effort level is 0.57624, *which is lower than* 0.6601, *the expected effort level when only a final review is conducted. It can easily be verified that the second-order conditions are satisfied as well.* $\triangleleft$

The main objective of the example above was to demonstrate that unless the allocation rule is chosen carefully, one might end up with a mechanism that is inferior to the one in which only one review is conducted. While the exact mechanism, and in particular the exact value of $g^*(1,2)$, varies with the different parameters of the problem, the following lemma shows that a two-review system, in which the midterm review is used only as a tie-breaking device for the final review, i.e., $g(1,2) = 0$, while not always optimal, nevertheless dominates the one-review system.

Lemma 3. *If* $g(1,2) = 0$, *then the expected effort in the two-review system is higher than that in the one-review system. That is,*

$$\overline{e}^1 + 2f_1(1)e^{LF} + (1 - 2f_1(1))e^T > \hat{e}^1 + \hat{e}^2.$$

Proof. Consider the two-review system and note that when $g(1,2) = 0$, then

$$e^T = e^{LF} = \bar{e}^2 \text{ where } c'(\bar{e}^2) = f_1'(0),$$

and the effort level in stage two is given by

$$c'(\bar{e}^1) = f_1'(0)[f_0(0) + \delta].$$

Next consider the equilibrium equation (5) for the one-review system. We conclude that (for all $f_0(0) > 0$) $\bar{e}^2 = \hat{e}^2$ and $\bar{e}^1 > \hat{e}^1$. □

Note that the lower $g(1,2)$ is, the higher the incentives are to exert effort in the second stage. However, lowering $g(1,2)$ decreases the incentives of the agents to exert effort in the first stage because it decreases the prize assigned to the winner of stage one**.** The optimal $g(1,2)$ exactly balances this trade-off. The following theorem characterizes $g^*(1,2)$ and in particular demonstrates that if $c''' > 0$, then $g^*(1,2)$ decreases as δ increases. In words, the more effective the first stage-effort is in determining the outcome of the final review, i.e., the higher δ is, the smaller is the weight that should be assigned to the midterm in determining the allocation of the prize. The intuition behind this result is fairly straightforward. When δ is small the midterm review is the more effective tool to get the agents to exert effort in the first stage. But assigning a high weight to the midterm review has an adverse effect on the second-stage effort. Thus, when δ gets larger, the first-stage effort has an effect on the final review's outcome, and agents exert effort in stage one even when the weight that is assigned to the midterm review is very low. Because decreasing the weight assigned to the midterm review increases the expected effort in stage two, it is optimal to do so.

Theorem 4. *If $c'''(\cdot) > 0$, then there exists $\delta^* \in (0,1)$ such that*

$$g^*(1,2) = \begin{cases} \frac{2f_1(0)}{1+2f_1(0)} & \text{if} \quad \delta = 0 \\ 0 & \text{if} \quad \delta \geq \delta^* \end{cases}.$$

For all other values of δ, $g^*(1,2)$is the solution to

$$c''(e^1) = (1-\delta)\,c''(e^{LF}).$$

Moreover, $g^*(1,2)$ decreases with $\delta$and δ^* is the solution to the equation

$$c''(e^1) = (1-\delta)\,c''(e^{LF})$$

for $g(1,2) = 0$.[8]

Proof. Recall that $g^*(1,2)$ is chosen to maximize

$$TE(g(1,2)) = \overline{e}^1(g(1,2)) + 2f_1(0)e^{LF}(g(1,2)) + (1-2f_1(0))e^T(g(1,2)).$$

It follows from (6) that $\frac{\partial e^T}{\partial g(1,2)} = 0$ and hence

$$\frac{\partial TE}{\partial g(1,2)} = \frac{\partial \overline{e}^1}{\partial g(1,2)} + 2f_1(0)\frac{\partial e^{LF}}{\partial g(1,2)}.$$

From (8)

$$\frac{\partial \bar{e}^1}{\partial g(1,2)} = -\frac{2f_1\left(0\right)f_1'\left(0\right)\left(1-\delta\right)}{-c''(e^1)} > 0$$

and from (7)

$$\frac{\partial e^{LF}}{\partial g(1,2)} = -\frac{-f_1'\left(0\right)}{-c''(e^{LF})}.$$

We get

$$\frac{\partial TE}{\partial g(1,2)} = \frac{\partial \bar{e}^1}{\partial g(1,2)} + 2f_1\left(0\right)\frac{\partial e^{LF}}{\partial g(1,2)} = 2f_1\left(0\right)f_1'\left(0\right)\left[\frac{1-\delta}{c''(e^1)} - \frac{1}{c''(e^{LF})}\right]. \tag{10}$$

First note that the assumed $c'''(\cdot) > 0$ guarantees that $\frac{\partial TE}{\partial g(1,2)} = 0$ is a point of maximum. Now, when $\delta = 0$, $\frac{\partial TE}{\partial g(1,2)} = 0$ implies that $\bar{e}^1 = e^{LF}$, which together with (7) and (8) yields

$$g^*(1,2) = \frac{2f_1(0)}{1+2f_1(0)}.$$

Next note that from (8) it follows that $\bar{e}^1$ increases with δ and $g(1,2)$ while e^{LF} decreases with $g(1,2)$. It is now easy to see from (10) that $g^*(1,2)$ decreases with δ. Finally, observe that when $\delta = 1$, $\frac{\partial TE}{\partial g(1,2)} < 0$ for any $g(1,2)$, and we conclude that in this case $g^*(1,2) = 0$. □

The economics behind the theorem

With the characterization of $g^*(s_1, s_2)$ in hand, it is instructive to go back and study the effort levels in the different mechanisms as derived by first-order conditions (5), (8), (7), and (6). Consider the effort level in stage two when only a final review is conducted and compare it to the effort level after a midterm review. Unlike when the midterm review results in a tie, the effort level is not the same in both mechanism when the midterm review results in a leader. In

[8] The results of the theorem remain qualitatively the same when the midterm and the final reviews are given by the different probability functions Γ^m and Γ^f, respectively, provided the final review process is more precise in the following sense. We say that the review process Γ^i is more precise than Γ^j if $f_1'^i(0) \geq f_1'^j(0)$ and $f_0^i(0) = f_0^j(0)$. That is, if both reviews have the same probability of tieing because equal efforts were exerted, the more precise process will detect with higher probability a deviation from the symmetric efforts.

particular, the effort level in stage two when only a final review is conducted is higher than the effort level in the subgame when there is a leader. The reason for this reduction in effort is clear. When only a final review is conducted, the stakes in the second stage are very high because the difference between winning and losing is the difference between receiving the prize with probability one and not receiving it (recall that in a symmetric equilibrium the agents exert the same effort level in stage one). Hence the incentive to invest in stage two is high. This is not the case in a subgame in which there is a leader, however. In particular, for agent one who is the leader the difference between winning and losing is $g^*(1,1) - g^*(1,2) \leq 1$, while if he is the follower it is $g^*(2,1) - g^*(2,2) \leq 1$.

Thus, the cost of conducting a midterm review is the reduction in effort in the second period. The gain from conducting a midterm review is in building incentives to exert effort in stage one. That is, the effort level in stage one is always higher when a midterm review is conducted, and it increases with the weight assigned to the midterm review, i.e., with $g^*(1,2)$. It follows that the designer, in choosing $g^*(1,2)$, must balance between these two opposite effects of $g^*(1,2)$. Now, because the effort level in stage one is increasing with δ and because $c''' > 0$, it follows that $g^*(1,2)$ decreases with δ. Indeed, it follows from (10) that when we have an interior solution for the optimal mechanism, i.e., $0 < g^*(1,2) < 1$, the equality

$$c''(e^1) = c''(e^{LF})\,(1-\delta) \tag{11}$$

must hold. In particular, $c''' > 0$ implies that $e^1 \leq e^{LF}$ and that an increase in δ must be followed by a decrease in $g^*(1,2)$. Once $g^*(1,2)$ becomes 0, any further increase in δ does not change $g(1,2)$ and henve it increases e^1. Therefore, for δ close enough to 1, $g(1,2) = 0$ and $e^1 > e^{LF}$.

The Rule of $f_0(\cdot)$

As mentioned, an important aspect of the current model is that agents' outputs are unobservable. This is in contrast to models in which outputs are observable and are determined by input plus a smooth noise. Indeed, we believe that ties are an inherent element in most, if not all, reviews, and the fact that we often observe reviews with a very fine grading system is really nothing but an arbitrary way of breaking a tie. For example, in the case of class-ranking tournaments among students, a grading system of 0-100 is probably too fine and an A,B,C,D grading might yield a more accurate picture, yet the 0-100 system can be seen as a tool for breaking a tie. Now, one implication of the analysis here is that in a multi review-system, breaking a tie in an arbitrary way is optimal only when both reviews result in a tie. In particular breaking a tie in the midterm independently of the final review is suboptimal.

Given that in most cases of interest $f_0(x) > 0$ for $x \in [-\varepsilon, +\varepsilon]$ and εsmall enough, it is worth noting that our main characterization result still holds for the case where $f_0(x) = 0$ for all x.In particular, when $f_0(x) = 0$ for all x and $\delta = 0,$ the two reviews are treated symmetrically and each receives equal weight. As δ increases the weight assigned to the midterm decreases monotonically and for all $\delta > \bar{\delta}$ all the weight is assigned to the final.

6 Endogenous prizes

Thus far we have considered the case where the designer has a fixed prize that has to be allocated to one of the players. While this setup, standard in the literature, nicely fits many situations of interest, it is inappropriate in many others. For example, when the tournament is among workers who compete to be promoted and receive a raise in their salary, the size of the prize is not fixed but rather a strategic decision of the designer and may vary according to the tournament's outcome. Thus, unlike the case of a fixed prize where the designer's goal was simply to maximize the agents' expected efforts, now when the prize is costly, he must balance the incentives to exert effort against the cost of the prize. In particular the designer's goal can be written as

$$\max E\left[\sum_i \sum_t e_i^t - payment\right],$$

and we restrict our attention to the case where only non-negative payments are allowed.

We start by studying a one stage tournament and identify conditions on Γ under which it is optimal to treat the agents symmetrically. To this end we add the following assumption on

$$\Gamma\left(e_1^1, e_2^1, e_1^2, e_2^2\right) = \left[f_1\left(\tau_1 - \tau_2\right),\ \ f_0\left(\tau_1 - \tau_2\right),\ \ f_2\left(\tau_1 - \tau_2\right)\right].$$

Ae: for any $x < 0$, with $f_0(x) > 0$

$$\frac{f_1'(x)}{f_1(x)} \geq \frac{f_0'(x)}{f_0(x)}. \tag{12}$$

Assumption **Ae** implies that for $x < 0$, $f_1(x)$ is more elastic than $f_0(x)$. That is, a small increase in effort on the part of a player who is lagging behind increases, precentage wise, his probability of winning more than his probability of tieing.

We then extend the analysis to the two-stage mechanism where it is shown that under the same conditions, the optimal design generates an asymmetric tournament in the second stage when the winner of the midterm review receives a favorable treatment in the second stage (and this holds even when the first-stage effort has no effect on second stage outcome (i.e., $\delta = 0$)). In particular, the marginal revenue from winning the final is higher for the winner of the midterm review than it is for the loser. As it turns out, the adverse effect on the incentives in the second stage due to the asymmetric treatment is more than compensated for by the gain in effort in the first round.

6.1 One Stage

Denote by w_i the expected payment to player i if he is announced to be the winner of the tournament, and similarly let l_i and t_i denote his payment when he is the loser or in case of a tie, respectively.

The expected utility of player i from choosing e^i, when his opponent chooses e^j, is given by

$$-c\left(e^i\right) + f_1\left(e^i - e^j\right) w_i + f_0\left(e^i - e^j\right) t_i + f_2\left(e^i - e^j\right) l_i.$$

Solving for first-order condition yields

$$c'\left(e^i\right) = \begin{cases} A_i & \text{if} \quad A_i \geq 0 \\ 0 & \text{if} \quad A_i < 0 \end{cases}, \tag{13}$$

where

$$A_i = f_1'\left(e^i - e^j\right) w_i + f_0'\left(e^i - e^j\right) t_i + f_2'\left(e^i - e^j\right) l_i.$$

Definition 1. *Given aggregate effort level E, the optimal mechanism, denoted by $F^*(E) : \mathcal{R}_+ \to \mathcal{R}_+^6$, assigns a vector of payments $(w_i^*, t_i^*, l_i^*)_{i=1,2}$ such that (i) the effort levels e^1 and e^2 given $(w_i^*, t_i^*, l_i^*)_{i=1,2}$ satisfy $e^1 + e^2 \geq E$ and (ii) for any vector of payments $(w_i', t_i', l_i')_{i=1,2}$ for which the total equilibrium effort is at least E, the expected payment is higher than in $(w_i^*, t_i^*, l_i^*)_{i=1,2}$.*

The following theorem establishes that the optimal mechanism is symmetric and only the winner gets a positive prize.

Theorem 5. *The optimal mechanism is symmetric and $(w_i^*, t_i^*, l_i^*)_{i=1,2} = (w, 0, 0)_{i=1,2}$.*

Proof. See Appendix C. □

The intuition behind the theorem is rather straightforward. Clearly, allocating positive prizes to the looser is suboptimal as it decreases incentives. Assumption Ae is sufficient for any prize, in case of a tie, to be suboptimal as well.

6.2 Two stages

We now extend the analysis of endogenous prizes to the two-stage mechanism in which a midterm review is conducted after the first stage.

Denote by $P_i\left(s_1, s_2\right)$ the prize to player $i \in \{1, 2\}$ when the results of the first and second rounds are s_1 and s_2, respectively, where $s_1, s_2 \in \{0, i, j\}$. We analyze the players' incentives in any symmetric mechanism and restrict our attention to the symmetric equilibrium. In a symmetric equilibrium agents choose the same effort level in the first stage but their effort in the second-stage might be different and depend on the first-stage review. Indeed, we show that in the second stage subgame, it is optimal to give a favorable treatment to the agent who came out first in the first-stage review. That is,

$$P_i\left(s_i, s_i\right) - P_i\left(s_i, s_j\right) \geq P_i\left(s_j, s_i\right) - P_i\left(s_j, s_j\right).$$

While this asymmetric treatment reduces incentives to exert effort in the second stage (a corollary of the main result in Section 6.1), it more than compensates for this by increasing incentives to exert effort in the first stage. As a result, the designer can pay less in expectations for a given level of effort.

We start by studying the equilibrium in the second-stage subgame.

The subgame after a tie in the first stage

Let e^T denote the equilibrium effort in the subgame after a tie in the first round. The expected utility of player i who chooses effort level e is given by

$$-c(e) + f_1(e - e^T)P_i(0, i) + f_0(e - e^T)P_i(0, 0) + f_2(e - e^T)P_i(0, j).$$

The first-order condition yields

$$-c'(e) + A \begin{cases} = 0 & \text{if} \quad A \geq 0 \\ < 0 & \text{if} \quad A < 0 \end{cases}$$

where

$$A = f_1'(e - e^T)\left(P_i(0, i) - P_i(0, j)\right) + f_0'(e - e^T)(P_i(0, 0) - P_i(0, j)).$$

In equilibrium both players choose effort level e^T satisfying

$$-c'(e^T) + f_1'(0)\left(P_i(0, i) - P_i(0, j)\right) \begin{cases} = 0 & \text{if} \quad P_i(0, i) - P_i(0, j) \geq 0 \\ < 0 & \text{if} \quad P_i(0, i) - P_i(0, j) < 0 \end{cases}. \tag{14}$$

The subgame after a win in the first stage

Let e^F denote the second-stage equilibrium effort level of the agent who lost the first-round review. The expected utility in the subgame of the player who won the first-stage review and who chooses effort level e in the second stage is given by

$$-c(e) + f_1(e - e^F)P_i(i, i) + f_0(e - e^F)P_i(i, 0) + f_2(e - e^F)P_i(i, j).$$

Solving for first-order condition yields

$$-c'(e^L)+B\begin{cases}=0 & \text{if} \quad B\geq 0\\ <0 & \text{if} \quad B<0\end{cases} \tag{15}$$

where

$$B=f_1'(e^L-e^F)\left(P_i(i,i)-P_i(i,j)\right)+f_0'(e^L-e^F)(P_i(i,0)-P_i(i,j))$$

Similarly,

$$-c'(e^F)+C\begin{cases}=0 & \text{if} \quad C\geq 0\\ <0 & \text{if} \quad C<0\end{cases} \tag{16}$$

where

$$C=f_1'(e^F-e^L)\left(P_i(j,i)-P_i(j,j)\right)+f_0'(e^F-e^L)(P_i(j,0)-P_i(j,j)).$$

We are now ready to back up and study the equilibrium effort in the first stage of the tournament.

First stage

Using the same notations and following the same logic as in Section 4 we derive the first-order condition for the first-stage effort e^1 as

$$-c'(e^1)+D\begin{cases}=0 & \text{if} \quad D\geq 0\\ <0 & \text{if} \quad D<0\end{cases} \tag{17}$$

where

$$\begin{aligned} D &= f_1'(0)\left(u^L-u^F\right)+\\ &\quad f_1(0)\left[\delta f_1'(e^L-e^F)\left(P_i(i,i)-P_i(i,j)\right)+\delta f_0'(e^L-e^F)(P_i(i,0)-P_i(i,j))\right]\\ &\quad +f_0(0)\,\delta f_1'(0)\left(P_i(0,i)-P_i(0,j)\right)+\\ &\quad f_2(0)\left[\delta f_1'(e^F-e^L)\left(P_i(j,i)-P_i(j,j)\right)+\delta f_0'(e^F-e^L)(P_i(j,0)-P_i(j,j))\right]\end{aligned}$$

and

$$\begin{gathered} u^L-u^F=c(e^F)-c(e^L)+f_1(e^L-e^F)\left[P_i(i,i)-P_i(j,j)\right]\\ +f_0(e^L-e^F)\left[P_i(i,0)-P_i(j,0)\right]+f_2(e^L-e^F)\left[P_i(i,j)-P_i(j,i)\right]. \end{gathered} \tag{18}$$

The principal's problem

The principal's goal is to induce the agents to choose the given expected effort at the minimal expected cost. In equilibrium, expected effort is given by

$$2e^1 + f_0(0)2e^T + (1 - f_0(0))\left(e^L + e^F\right) \tag{19}$$

while the expected cost is given by

$$\begin{aligned} &f_0(0)\left[f_0(0)2P_i(0,0) + (1 - f_0(0))\left(P_i\left(0,i\right) + P_i\left(0,j\right)\right)\right] + \\ &(1 - f_0(0))\left[f_0(e^L - e^F)\left(P_i(i,0) + P_i(j,0)\right) + f_1(e^L - e^F)\left(P_i(i,i) + P_i(j,j)\right)\right. \\ &\left.+ f_2(e^L - e^F)\left(P_i(i,j) + P_i(j,i)\right)\right]. \end{aligned} \tag{20}$$

The main result

Denote by $B(s_1, s_2) = P_1(s_1, s_2) + P_2(s_1, s_2)$the sum of the prizes to be allocated when the reviews result in $s_1, s_2 \in \{0, i, j\}$. We are now ready to state the main theorem of this section, which characterizes the optimal prizes for all $\Gamma = (f_1, f_0, f_2)$ that fall within some given set of parameters (defined precisely below)[9]. In particular, if for all x, $f_1''(x)$ is bounded above by some positive number small enough, then Γ is in the set. Alternatively, if for all x the function $f_1(x)/\left(f_1'(x)\right)^2$ is increasing, then Γ is in the set. Therefore, the set includes all review processes where the marginal winning probability either decreases or does not increase too quickly.

Theorem 6. *Assume that $\delta < 1$ and that for any $x > 0$ we have*

$$(a)\;\; 1 - 2\frac{f_1''(x)f_1(x)}{\left(f_1'(x)\right)^2} \geq 0 \;\; \textit{and} \;\; (b)\; 1 - \frac{f_1''(x)f_1(x)}{\left(f_1'(x)\right)^2} - \frac{f_1''(-x)f_1(-x)}{\left(f_1'(-x)\right)^2} \geq 0.$$

Then in the optimal mechanism

$$\begin{aligned} (i) \quad\quad e^L &> e^F \\ (ii) \quad B(i,i) &> B(i,j) \\ (iii) \quad B(i,j) &\geq B(i,0) \geq B(0,0) \end{aligned}$$

where $i, j \in \{1, 2\}$.

Proof. See Appendix E. As is apparent from the proof, the theorem also holds for the case in which $f_0(x) = 0$ for all x. □

Since without a midterm review both agents exert equal effort levels, it follows from Theorem (6) that conducting a midterm review is always optimal for the principal.

[9] The result still holds if the ranking probabilities for the midterm and the final review processes are different. In this case, the conditions on Γ given in the theorem above refer to the final review process only.

Corollary 1. *The expected utility of the principal in the optimal mechanism with a midterm review is strictly higher than in the mechanism with a final review only.*

The theorem establishes that in the second stage it pays to offer an incentive to the leader to exert greater effort compared to the effort exerted by the follower. Recall that the expected payment in a one-stage tournament is minimized when both agents exert the same effort level. Let us now consider a small increase in the prize to the first-round winner should he also win the second-round, and a small decrease in the prize to the first-round loser should he win the second round, such that the second-stage aggregate effort remains unchanged. Such a change has two effects; on the one hand the incentives to exert effort in the first stage are stronger because the first-stage winner will compete at the second-stage for the higher prize[10]. Moreover, his opponent at the second stage will exert less effort, which increases his chances of winning the second stage. On the other hand, such a change increases the expected payment to the agents. However, the effect of the change on the expected payments is only of the second order and can be completely offset by decreasing the payment to the first-round winner should he lose the second round. This intuition and, in particular, the fact that $B(i,i) > B(i,j)$ are what is behind the ranking of the payments . As for the case of a fixed prize, a positive payment after a tie has no effect on incentives, which explains the rest of the ranking. In Appendix E we show by example that the ranking of $B(i,j)$ and $B(0,i)$ is ambiguous and indeed can go either way.

Concluding Remark: While we have restricted ourselves to the case of two reviews only, we believe that the qualitative results as well as the intuition developed here extend to the case of n reviews. We leave this, however, for future research.

7 Appendix

7.1 A. Proof of Lemma 1

We are interested in showing that $\overline{e}^1(g(s_1,s_2))$, $e^{LF}(g(s_1,s_2))$, and $e^T(g(s_1,s_2))$ solve agent i's first-order condition system of equations, when agent j's effort levels are set to $\overline{e}^1(g(s_1,s_2))$, $e^{LF}(g(s_1,s_2))$ and $e^T(g(s_1,s_2))$. We start by deriving the FOC in the subgames following the midterm review. Consider the subgame following a tie. First note that although the midterm review resulted in a tie, the player that invested more in the first stage has an advantage since it increases his probability of winning the final review. Let $\Delta^T(i,j) = \delta e_i^1 + e_i^T - \delta e_j^1 - e_j^T$, where e_j^T is the effort level of agent j in the subgame. Agent i's expected utility when his effort level is e_i^T is

$$-c(e_i^T) + f_1\left(\Delta^T(i,j)\right)g(0,1) + f_0\left(\Delta^T(i,j)\right)g(0,0) + f_2\left(\Delta^T(i,j)\right)g(0,2). \qquad (21)$$

[10] Conditions (a) and (b) of Theorem 6 are sufficient for this effect to take place.

Recall that $f_2(x) = 1 - f_1(x) - f_0(x)$, and rewrite (21) as

$$-c(e_i^T) + f_1(\Delta^T(i,j))\,(g(0,1) - g(0,2)) + f_0(\Delta^T(i,j))\,(g(0,0) - g(0,2)) + g(0,2).$$

Agent i's first-order condition is

$$c'(e_i^T) = \Big[f_1'(\Delta^T(i,j))\,(g(0,1) - g(0,2)) + f_0'(\Delta^T(i,j))\,(g(0,0) - g(0,2)) \Big]. \tag{22}$$

Finally, note that if both agents exert the same effort level in stage one, then there exists a symmetric solution $e_1^T = e_2^T = e^T(g(s_1, s_2))$ that solves the first-order condition (22) for both agents for which

$$c'(e^T) = f_1'(0)\,(g(0,1) - g(0,2))$$

if $g(0,1) - g(0,2) > 0,$ and $e^T = 0$ otherwise, as stated in the lemma..

Before moving to the subgame in which there is a leader, we note that the utility of agent i in this subgame is given by

$$\begin{aligned} u_i^T(e_i^1, e_i^T(e_i^1)) = -c(e_i^T(e_i^1)) + f_1\left(\widetilde{\Delta}^T(i,j)\right)(g(0,1) - g(0,2)) \\ + f_0\left(\widetilde{\Delta}^T(i,j)\right)(g(0,0) - g(0,2)) + g(0,2) \end{aligned} \tag{23}$$

where $e_i^T(e_i^1)$ is a solution of the first-order condition, and

$$\widetilde{\Delta}^T(i,j) = \delta e_i^1 + e_i^T(e_i^1) - \delta e_j^1 - e_j^T(e_j^1).$$

Assume agent i was ranked first in the midterm review and is now the leader. Using **As**, agent i's expected utility in the subgame when his effort level is e_i^L and his rival's effort level is e_j^F is

$$-c(e_i^L) + f_1(\Delta^L(i,j))\,(g(1,1) - g(1,2)) + f_0(\Delta^L(i,j))\,(g(1,0) - g(1,2)) + g(1,2)$$

where, as before, $\Delta^L(i,j) = \delta e_i^1 + e_i^L - \delta e_j^1 - e_j^F$.

The leader's first-order condition is

$$c'(e_i^L) = f_1'(\Delta^L(i,j))\,(g(1,1) - g(1,2)) + f_0'(\Delta^L(i,j))\,(g(1,0) - g(1,2)). \tag{24}$$

Similarly, we can express the follower's expected utility and the corresponding first-order condition as

$$-c(e_j^F) + f_1(-\Delta^L(i,j))\,(g(1,1) - g(1,2)) + f_0(-\Delta^L(i,j))(g(1,1) - g(1,0))$$
$$+(1 - g(1,1))$$

and

$$c'(e_j^F) = f_1'(-\Delta^L(i,j))\,(g(1,1) - g(1,2)) + f_0'(-\Delta^L(i,j))(g(1,1) - g(1,0)). \tag{25}$$

As in the subgame following a tie, if both players choose the same effort in stage one, there exists a symmetric solution $e_1^L = e_2^F = e^{LF}(g(s_1, s_2))$ which solves the two first-order conditions (24) and (25) and satisfies

$$c'(e^{LF}) = f_1'(0)\,(g(1,1) - g(1,2))$$

if $g(1,1) - g(1,2) > 0$ and $e^{LF} = 0$ otherwise, as stated in the lemma.

It follows that the utilities of the agents in the subgame are given by

$$u_i^L(e_i^1, e_i^L(e_i^1)) = -c(e_i^L(e_i^1)) + f_1\left(\widetilde{\Delta}^L(i,j)\right)[g(1,1) - (g(1,2)] \tag{26}$$
$$+f_0\left(\widetilde{\Delta}^L(i,j)\right)[g(1,0) - (g(1,2)] + g(1,2)$$

and

$$u_j^F(e_j^1, e_j^F(e_j^1)) = -c(e_j^F(e_j^1)) + f_1\left(-\widetilde{\Delta}^L(i,j)\right)[g(1,1) - g(1,2)] \tag{27}$$
$$+f_0\left(-\widetilde{\Delta}^L(i,j)\right)[g(1,1) - g(1,0)] + 1 - g(1,1)$$

where $e_i^L(e_i^1)$ and $e_j^F(e_j^1)$ are the solutions to the system of two corresponding first-order conditions (24) and (25) and $\widetilde{\Delta}^L(i,j) = \delta e_i^1 + e_i^L(e_i^1) - \delta e_j^1 - e_j^F(e_j^1)$.

Assuming now that agent i's effort levels in the subgames are $e^{LF}(g(s_1, s_2))$ and $e^T(g(s_1, s_2))$, it is left for us to show that

$$\begin{aligned} c'(\overline{e}^1) \;\; = & \\ & f_1'(0)\,[2f_1(0)\,(g(1,1) + g(1,2) - 1) + f_0(0)\,(2g(1,0) - 1)] \\ & +2f_1(0)\,\delta f_1'(0)\,[g(1,1) - g(1,2)] \\ & +f_0(0)\,\delta f_1'(0)\,[g(0,1) - g(0,2)] \end{aligned}$$

if the expression on the right-hand side of the previous equality is positive. Recall that agent i's expected utility in the mechanism is

$$-c(e_i^1) + f_1\left(e_i^1 - e_j^1\right) u_i^L(e_i^1, e_i^L(e_i^1)) + f_0\left(e_i^1 - e_j^1\right) u_i^T(e_i^1, e_i^T(e_i^1))$$
$$+ f_2\left(e_i^1 - e_j^1\right) u_i^F(e_i^1, e_i^F(e_i^1)),$$

which yields the following first-order condition with respect to e_i^1 :

$$\begin{aligned} c'(e_i^1) &= f_1'\left(e_i^1 - e_j^1\right) u_i^L(e_i^1, e_i^L(e_i^1)) + f_1\left(e_i^1 - e_j^1\right) \frac{du_i^L(e_i^1, e_i^L(e_i^1))}{de_i^1} \\ &\quad + f_0'\left(e_i^1 - e_j^1\right) u_i^T(e_i^1, e_i^T(e_i^1)) + f_0\left(e_i^1 - e_j^1\right) \frac{du_i^T(e_i^1, e_i^T(e_i^1))}{de_i^1} \\ &\quad + f_2'\left(e_i^1 - e_j^1\right) u_i^F(e_i^1, e_i^F(e_i^1)) + f_2\left(e_i^1 - e_j^1\right) \frac{du_i^F(e_i^1, e_i^F(e_i^1))}{de_i^1}. \end{aligned} \tag{28}$$

Using $f_2'(x) = -f_1'(x) - f_0'(x)$ we can rewrite (28) as

$$c'(e_i^1) = f_1'\left(e_i^1 - e_j^1\right)\left[u_i^L(e_i^1, e_i^L(e_i^1)) - u_i^F(e_i^1, e_i^F(e_i^1))\right]$$
$$+ f_0'\left(e_i^1 - e_j^1\right)\left[u_i^T(e_i^1, e_i^T(e_i^1)) - u_i^F(e_i^1, e_i^F(e_i^1))\right] + f_1\left(e_i^1 - e_j^1\right) \frac{du_i^L(e_i^1, e_i^L(e_i^1))}{de_i^1}$$
$$+ f_0\left(e_i^1 - e_j^1\right) \frac{du_i^T(e_i^1, e_i^T(e_i^1))}{de_i^1} + f_2\left(e_i^1 - e_j^1\right) \frac{du_i^F(e_i^1, e_i^F(e_i^1))}{de_i^1}.$$

Next note that for $Z \in \{T, L, F\}$ and for $i = 1, 2$,

$$\frac{du_i^Z(e_i^1, e_i^Z(e_i^1))}{de_i^1} = \frac{\partial u_i^Z(e_i^1, e_i^Z(e_i^1))}{\partial e_i^1} + \frac{\partial u_i^Z(e_i^1, e_i^Z(e_i^1))}{\partial e_i^Z(e_i^1))} \frac{\partial e_i^Z(e_i^1))}{\partial e_i^1}.$$

However, since $e_i^Z(e_i^1))$ maximizes $u_i^Z(e_i^1, e_i^Z(e_i^1))$ for any e_i^1, we have[11]

$$\begin{aligned} &\frac{du_i^L(e_i^1, e_i^L(e_i^1))}{de_i^1} \\ &= \delta\left[f_1'\left(\widetilde{\Delta}^L(i,j)\right)(g(1,1) - g(1,2)) + f_0'\left(\widetilde{\Delta}^L(i,j)\right)(g(1,0) - g(1,2))\right], \end{aligned}$$

$$\begin{aligned} &\frac{du_i^F(e_i^1, e_i^F(e_i^1))}{de_i^1} \\ &= \delta\left[f_1'\left(-\widetilde{\Delta}^L(j,i)\right)(g(2,1) - g(2,2)) + f_0'\left(-\widetilde{\Delta}^L(j,i)\right)(g(2,0) - g(2,2))\right], \end{aligned}$$

[11] Since $e_i^Z(e_i^1))$ maximizes $u_i^Z(e_i^1, e_i^Z(e_i^1))$, either $\frac{\partial u_i^Z(e_i^1, e_i^Z(e_i^1))}{\partial e_i^Z(e_i^1))} = 0$ or $e_i^Z(e_i^1)) = 0$. Therefore, the second term in the last expression is 0.

and

$$\begin{aligned}
&\frac{du_i^T(e_i^1, e_i^T(e_i^1))}{de_i^1} \\
= \; &\delta \left[f_1' \left(\widetilde{\Delta}^T(i,j) \right) (g(0,1) - g(0,2)) + f_0' \left(\widetilde{\Delta}^T(i,j) \right) (g(0,0) - g(0,2)) \right].
\end{aligned}$$

Now, if in every second-stage subgame both agents exert the same effort (i.e., $e^{LF}(g(s_1, s_2))$ and $e^T(g(s_1, s_2))$), then both agents have the same first-order conditions determining first-stage effort. This implies that there exists a solution to the first-order condition (28) in which both agents choose the same effort in stage one. Hence, stage two's first-order conditions are indeed given by (6) and (7) and $\widetilde{\Delta}^Z(i,j) = 0$ for $Z \in \{T, L, F\}$. Moreover, from (26) and (27) it follows that

$$\begin{aligned}
&u_i^L(e_i^1, e_i^L(e_i^1) - u_i^F(e_i^1, e_i^F(e_i^1)) \\
= \; &f_0(0)\left[2g(1,0) - g(1,2) - g(1,1)\right] - g(2,2) + g(1,2) \\
= \; &f_0(0)\left[2g(1,0) - 1\right] + 2f_1(1)\left[g(1,2) - 1 + g(1,1)\right].
\end{aligned}$$

Plugging the last expressions into (28) we get, using $f_1'(0) = -f_2'(0)$ and $f_0'(0) = 0$, the required equality (8), as stated in the lemma.

To complete the proof, we have to show the uniqueness of the symmetric solution to the first-order conditions. Clearly, the solution is unique for $\overline{e}^1$ and e^T. Therefore, it is enough to show that there is a unique solution to the first-order condition in the case where there is a leader. Recall that if at the first stage both agents exert the same effort, then the first-order conditions (25) and (24) boil down to

$$\begin{aligned}
c'(e_i^L) &= f_1'(e^L - e^F)\,(g(1,1) - g(1,2)) + f_0'(e^L - e^F)\,(g(1,0) - g(1,2)) \\
c'(e_j^F) &= f_1'(e^F - e^L)\,(g(1,1) - g(1,2)) + f_0'(e^F - e^L)(g(1,1) - g(1,0))
\end{aligned}$$

Notice first that As implies that $f_1'(x) = -f_2'(-x)$. Moreover,

$$f_1'(x) + f_0'(x) + f_2'(x) = 0$$

implies that

$$f_1'(x) = f_1'(-x) - f_0'(x).$$

Therefore, we can rewrite the first-order conditions for the follower in the following way:

$$\begin{aligned} c'(e_j^F) &= f_1'(e^L - e^F)\left(g(1,1) - g(1,2)\right) + f_0'(e^F - e^L)(g(1,1) - g(1,0) - g(1,1) + g(1,2)) \\ &= f_1'(e^L - e^F)\left(g(1,1) - g(1,2)\right) + f_0'(e^L - e^F)\left(g(1,0) - g(1,2)\right) \end{aligned}$$

where the second line follows from As, which implies that $f_0'(x) = -f_0'(-x)$**.** Therefore, $c'(e_j^F) = c'(e_i^L)$. Monotonicity of $c'(\cdot)$ completes the proof.■

7.2 B. Second-Order Conditions for Two Stages and a Fixed Prize

Lemma 7. *Assume there exists $\lambda > 0$, such that for any $x \in R$ the following holds:*

$$f_1^{'}(x), f_0'(x), \left|f_0''(x)\right|, \left|f_1''(x)\right|, \left|f_2''(x)\right| < \lambda.$$

If for any $y \in R_+$

$$c''(y) > 5\delta^2\lambda + 5\delta\lambda^2 + 3\lambda$$

then in the one-review system as well as the two-review system, the solution to the first-order conditions solves the agent maximization problem. That is, the second-order conditions for maximization hold.

Before proving the statement of Lemma (7) we first show that increasing the effort level after a tie in the first stage always increases the agent's probability of winning the prize.

Claim 1: For any $x \in R$, $f_1^{'}(x) + \frac{1}{2}f_0'(x) > 0$.

Proof. Observe first that for $x \leq 0$, the statement of the claim follows directly from **Ai**. Also note that **As** implies that

$$f_1(x) + \frac{1}{2}f_0(x) + f_1(-x) + \frac{1}{2}f_0(-x) = 1.$$

Differentiating with respect to x yields

$$f_1^{'}(x) + \frac{1}{2}f_0'(x) = f_1^{'}(-x) + \frac{1}{2}f_0'(-x),$$

which establishes the claim. □

We first prove the statement for the one-review system where only a final review is conducted.

Proof of Lemma 7 for the one-review system. Without loss of generality we restrict our attention to agent 1. Denote by $u_1(e^1, e^2)$ the expected utility of agent 1 in the second stage if his effort levels are e^1 and e^2, while his opponent plays $\overline{e}^1$,$\overline{e}^2$. Note that

$$\frac{\partial u_1(e^1,e^2)}{\partial e^2} = -c'(e^2) + \left[f_1' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) + \frac{1}{2} f_0' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) \right]. \tag{29}$$

Since $c'(0) = 0$, **Claim 1** implies that $c'(0) < f_1^{'}(x) + \frac{1}{2}f_0'(x)$ for any $x \in R$. Moreover, $c'' > \frac{3}{2}\lambda \geq f_1''(x) + \frac{1}{2}f_0''(x)$ implies that for any e^1 there exists a unique positive solution to $\frac{\partial u_1(e^1,e^2)}{\partial e^2} = 0$ that maximizes $u_1(e^1,e^2)$, and we shall denote this solution by $e^2(e^1)$. From the implicit function theorem it follows that

$$\frac{de^2(e^1)}{de^1} = -\delta \frac{f_1'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) + \frac{1}{2} f_0'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right)}{-c''(e^2) + f_1'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) + \frac{1}{2} f_0'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right)}.$$

Since $c''(e^2) > 3\lambda \geq 2 \left(f_1'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) + \frac{1}{2} f_0'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) \right)$, we can conclude that $\left| \frac{de^2(e^1)}{de^1} \right| < \delta$. Taking a derivative of agent 1's expected utility with respect to e^1 yields

$$-c'(e^1) + \frac{\partial u_1(e^1,e^2(e^1))}{\partial e^1} + \frac{\partial u_1(e^1,e^2(e^1))}{\partial e^2} \frac{de^2(e^1)}{de^1}.$$

The second derivative with respect to e^1 is given by

$$\begin{gathered}
-c''(e^1) + \frac{\partial^2 u_1(e^1,e^2(e^1))}{\partial \left(e^1\right)^2} + 2\frac{\partial^2 u_1(e^1,e^2(e^1))}{\partial e^2 \partial e^1} \frac{de^2(e^1)}{de^1} + \\
\frac{\partial u_1(e^1,e^2(e^1))}{\partial e^2} \frac{d^2 e^2(e^1)}{d\left(e^1\right)^2} + \frac{\partial^2 u_1(e^1,e^2(e^1))}{\partial \left(e^2\right)^2} \left(\frac{de^2(e^1)}{de^1} \right)^2.
\end{gathered}$$

In the sequel we shall show that the above expression is negative. Since $e^2(e^1)$ maximizes $u_1(e^1,e^2)$, it is enough to show that

$$-c''(e^1) + \frac{\partial^2 u_1(e^1,e^2(e^1))}{\partial \left(e^1\right)^2} + 2\frac{\partial^2 u_1(e^1,e^2(e^1))}{\partial e^2 \partial e^1} \frac{de^2(e^1)}{de^1} < 0. \tag{30}$$

Starting with the second term in (30), it follows from $f_Z'' < \lambda$ for $Z \in \{0,1\}$ that

$$\begin{aligned}
& \frac{\partial^2 u_1(e^1,e^2(e^1))}{\partial \left(e^1\right)^2} \\
= \;& \delta^2 \left(f_1'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) + \frac{1}{2} f_0'' \left(\delta e^1 + e^2 - \delta \overline{e}^1 - \overline{e}^2 \right) \right) \leq \frac{3}{2}\delta^2 \lambda.
\end{aligned}$$

The third term in (30)

$$
\begin{aligned}
&\frac{\partial^2 u_1(e^1, e^2(e^1))}{\partial e^2 \partial e^1}\frac{de^2(e^1)}{de^1} \\
&= \delta\left(f_1''\left(\delta e^1 + e^2 - \delta\overline{e}^1 - \overline{e}^2\right) + \frac{1}{2}f_0''\left(\delta e^1 + e^2 - \delta\overline{e}^1 - \overline{e}^2\right)\right)\frac{de^2(e^1)}{de^1} < \frac{3}{2}\delta^2\lambda
\end{aligned}
$$

Since $c'' > 5\delta^2\lambda + 5\delta\lambda^2 + 3\lambda$, we conclude that the second-order condition is satisfied. $\square$

We are now ready to prove the statement of the lemma for the two-review system, in which both the midterm review and the final review are conducted.

Proof of Lemma 7 for the two-review system. As before, we restrict our attention to agent 1 only. Assume that the opponent plays $\bar{e}^1$, $\bar{e}^T$, and $\bar{e}^{LF}$. Then

$$
\begin{aligned}
\frac{\partial u^L(e^1, e^L)}{\partial e^L} &= -c'(e^L) - f_2'(\delta e^1 + e^L - \delta\overline{e}^1 - \overline{e}^{LF})\,(1 - g(1,2)) \\
\frac{\partial u^T(e^1, e^T)}{\partial e^T} &= -c'(e^T) + f_1'(\delta e^1 + e^T - \delta\overline{e}^1 - \overline{e}^T) + \frac{1}{2}f_0'(\delta e^1 + e^T - \delta\overline{e}^1 - \overline{e}^T) \\
\frac{\partial u^F(e^1, e^F)}{\partial e^F} &= -c'(e^F) + f_1'(\delta e^1 + e^F - \delta\overline{e}^1 - \overline{e}^{LF})\,(1 - g(1,2))\,.
\end{aligned}
\tag{31}
$$

Proceeding along the same lines as before, since $c'(0) = 0$, we get that $\frac{\partial u^Z(e^1,0)}{\partial e^Z} > 0$ for any $Z \in \{L, F, T\}$. As before, $c'' > \frac{3}{2}\lambda \geq f_1''(x) + \frac{1}{2}f_0''(x)$ implies that for any e^1 there exists a unique positive solution to $\frac{\partial u^Z(e^1,e^Z)}{\partial e^Z} = 0$ that maximizes $u^Z(e^1, e^Z)$, which will be denoted by $e^Z(e^1)$. Similarly to the one-review system it can be shown that for any $Z \in \{L, F, T\}$

$$
\left|\frac{de^Z(e^1)}{de^1}\right| < \delta.
$$

Taking the derivative of agent 1's expected utility with respect to e^1 we obtain

$$
\begin{aligned}
&-c'(e^1) + f_1'\left(e^1 - \overline{e}^1\right)u^L(e^1, e^L(e^1)) + f_1\left(e^1 - \overline{e}^1\right)\frac{du^L(e^1, e^L(e^1))}{de^1} \\
&+f_0'\left(e^1 - \overline{e}^1\right)u^T(e^1, e^T(e^1)) + f_0\left(e^1 - \overline{e}^1\right)\frac{du^T(e^1, e^T(e^1))}{de^1} \\
&+f_2'\left(e^1 - \overline{e}^1\right)u^F(e^1, e^F(e^1)) + f_2\left(e^1 - \overline{e}^1\right)\frac{du^F(e^1, e^F(e^1))}{de^1}
\end{aligned}
$$

where

$$
\frac{du^Z(e^1, e^Z(e^1))}{de^1} = \frac{\partial u^Z(e^1, e^Z(e^1))}{\partial e^1} + \frac{\partial u^Z(e^1, e^Z(e^1))}{\partial e_1^Z}\frac{de^Z(e^1)}{de^1}.
$$

The second derivative is

$$
\begin{aligned}
&-c''(e^1) + f_1''\left(e^1 - \overline{e}^1\right) u^L(e^1, e^L(e^1)) + f_0''\left(e^1 - \overline{e}^1\right) u^T(e^1, e^T(e^1)) \\
&+ f_2''\left(e^1 - \overline{e}^1\right) u^F(e^1, e^F(e^1)) + 2f_1'\left(e^1 - \overline{e}^1\right) \frac{du^L(e^1, e^L(e^1))}{de^1} \\
&+ 2f_0'\left(e^1 - \overline{e}^1\right) \frac{du^T(e^1, e^T(e^1))}{de^1} + 2f_2'\left(e^1 - \overline{e}^1\right) \frac{du^F(e^1, e^F(e^1))}{de^1} \\
&+ f_1\left(e^1 - \overline{e}^1\right) \frac{d^2u^L(e^1, e^L(e^1))}{d\left(e^1\right)^2} + f_0\left(e^1 - \overline{e}^1\right) \frac{d^2u^T(e^1, e^T(e^1))}{d\left(e^1\right)^2} \\
&+ f_2\left(e^1 - \overline{e}^1\right) \frac{d^2u^F(e^1, e^F(e^1))}{d\left(e^1\right)^2}.
\end{aligned} \tag{32}
$$

In the sequel we shall show that for any strategy ($\overline{e}^1$,$\overline{e}^T$, $\overline{e}^{LF}$) of agent two, the expression in (32) is always negative.

First note that

$$
\begin{aligned}
\frac{d^2u^Z(e^1, e_1^Z(e^1))}{d\left(e^1\right)^2} &= \frac{\partial^2u^Z(e^1, e_1^Z(e^1))}{\partial\left(e^1\right)^2} + 2\frac{\partial^2u^Z(e^1, e_1^Z(e^1))}{\partial e^1 \partial e_1^Z} \frac{de_1^Z(e^1)}{de^1} \\
&+ \frac{\partial u^Z(e^1, e_1^Z(e^1))}{\partial e_1^Z} \frac{d^2e_1^Z(e^1)}{d\left(e^1\right)^2} + \frac{\partial^2u^Z(e^1, e_1^Z(e^1))}{\partial\left(e_1^Z\right)^2} \left(\frac{de_1^Z(e^1)}{de^1}\right)^2.
\end{aligned}
$$

However, since for any e^1, e_1^Z maximizes $u^Z(e^1, e_1^Z(e^1))$, we have that

$$
\frac{d^2u^Z(e^1, e_1^Z(e^1))}{d\left(e^1\right)^2} < \frac{\partial^2u^Z(e^1, e_1^Z(e^1))}{\partial\left(e^1\right)^2} + 2\frac{\partial^2u^Z(e^1, e_1^Z(e^1))}{\partial e^1 \partial e_1^Z} \frac{de_1^Z(e^1)}{de^1}. \tag{33}
$$

Next note that $u^Z(e^1, e^Z(e^1)) \in [0, 1]$, for any $Z \in \{L, F, T\}$, because non-negative utility at any subgame is guaranteed by $e_1^Z = 0$ and since the prize is 1, the utility cannot be higher than 1. Therefore,

$$
\begin{gathered}
f_1''\left(e^1 - \overline{e}^1\right) u^L(e^1, e^L(e^1)) + f_0''\left(e^1 - \overline{e}^1\right) u^T(e^1, e^T(e^1)) \\
+ f_2''\left(e^1 - \overline{e}^1\right) u^F(e^1, e^F(e^1)) < \\
\max\left\{f_1''\left(e^1 - \overline{e}^1\right), 0\right\} + \max\left\{f_0''\left(e^1 - \overline{e}^1\right), 0\right\} + \max\left\{f_2''\left(e^1 - \overline{e}^1\right), 0\right\} \leq 3\lambda.
\end{gathered} \tag{34}
$$

Recall that

$$
\frac{du^Z(e^1, e^Z(e^1))}{de^1} = \frac{\partial u^Z(e^1, e^Z(e^1))}{\partial e^1} + \frac{\partial u^Z(e^1, e^Z(e^1))}{\partial e_1^Z} \frac{de^Z(e^1)}{de^1} = \frac{\partial u^Z(e^1, e^Z(e^1))}{\partial e^1}
$$

where the last equality follows from the envelope theorem. Therefore,

$$\begin{aligned}
\frac{du^L(e^1,e^L(e^1))}{de^1} &= \delta\left(-f_2'\left(\delta e^1+e^L-\delta\overline{e}^1-\overline{e}^{LF}\right)\right)(1-g(1,2)) \\
\frac{du^T(e^1,e^T(e^1))}{de^1} &= \delta\left[f_1'(\delta e^1+e^T-\delta\overline{e}^1-\overline{e}^T)+\frac{1}{2}f_0'(\delta e^1+e^T-\delta\overline{e}^1-\overline{e}^T)\right] \\
\frac{du^F(e^1,e^F(e^1))}{de^1} &= \delta\left[f_1'(\delta e^1+e^F-\delta\overline{e}^1-\overline{e}^{LF})(1-g(1,2))\right].
\end{aligned}$$

Since $f_2'<0$, we can conclude that

$$\begin{aligned}
& 2f_1'\left(e^1-\overline{e}^1\right)\frac{du^L(e^1,e^L(e^1))}{de^1}+2f_0'\left(e^1-\overline{e}^1\right)\frac{du^T(e^1,e^T(e^1))}{de^1} \\
& +2f_2'\left(e^1-\overline{e}^1\right)\frac{du^F(e^1,e^F(e^1))}{de^1} \\
\leq\; & 2\delta\lambda^2+3\delta\lambda^2=5\delta\lambda^2.
\end{aligned} \tag{35}$$

From (33) it follows that

$$\begin{aligned}
\frac{d^2u^L(e^1,e_1^L(e^1))}{d\left(e^1\right)^2} &< \left(-\delta^2-2\delta\frac{de_1^L(e^1)}{de^1}\right)f_2''\left(\delta e^1+e^L-\delta\overline{e}^1-\overline{e}^{LF}\right)(1-g(1,2)) \\
&< 3\delta^2\left|f_2''\left(\delta e^1+e^L-\delta\overline{e}^1-\overline{e}^{LF}\right)\right|<3\delta^2\lambda.
\end{aligned}$$

Similarly, we get that

$$\begin{aligned}
\frac{d^2u^T(e^1,e_1^T(e^1))}{d\left(e^1\right)^2} &< \left(\delta^2+2\delta\frac{de_1^T(e^1)}{de^1}\right)\left(f_1''(\delta e^1+e^T-\delta\overline{e}^1-\overline{e}^T)+\frac{1}{2}f_0''(\delta e^1+e^T-\delta\overline{e}^1-\overline{e}^T)\right) \\
&< 3\delta^2\left(\left|f_1''(\delta e^1+e^T-\delta\overline{e}^1-\overline{e}^T)\right|+\frac{1}{2}\left|f_0''(\delta e^1+e^T-\delta\overline{e}^1-\overline{e}^T)\right|\right)<3\delta^2\frac{3}{2}\lambda
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2u^F(e^1,e_1^F(e^1))}{d\left(e^1\right)^2} &< \left(\delta^2+2\delta\frac{de_1^F(e^1)}{de^1}\right)(1-g(1,2))\,f_1''(\delta e^1+e^F-\delta\overline{e}^1-\overline{e}^{LF}) \\
&< 3\delta^2\left|f_1''(\delta e^1+e^F-\delta\overline{e}^1-\overline{e}^{LF})\right|<3\delta^2\lambda.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& f_1\left(e^1-\overline{e}^1\right)\frac{d^2u^L(e^1,e_1^L(e^1))}{d\left(e^1\right)^2}+f_0\left(e^1-\overline{e}^1\right)\frac{d^2u^T(e^1,e_1^T(e^1))}{d\left(e^1\right)^2} \\
& +f_2\left(e^1-\overline{e}^1\right)\frac{d^2u^F(e^1,e_1^F(e^1))}{d\left(e^1\right)^2} \\
\leq\ & 3\delta^2\frac{3}{2}\lambda.
\end{aligned} \tag{36}$$

In sum, since the second line in (32) is less than 3λ, and the third line is less than $5\delta\lambda^2$, and the last line is less than $3\delta^2\frac{3}{2}\lambda$, it follows that if

$$c'' > 5\delta^2\lambda + 5\delta\lambda^2 + 3\lambda,$$

then the second-order condition is satisfied. □

7.3 C. One-Stage, Endogenous Prizes

We prove Theorem 5 step by step by the sequence of 4 Claims.

Claim 1. *In the optimal mechanism*

$$c'\left(e^i\right) = A_i = f_1'\left(e^i-e^j\right)w_i^* + f_0'\left(e^i-e^j\right)t_i^* + f_2'\left(e^i-e^j\right)l_i^* \qquad i \neq j = 1, 2.$$

Proof. Assume by way of contradiction that for some i, $A_i < 0$. For this to hold, it must be the case that $l_i^* > 0$ (note that $e^i = 0$). Consider decreasing l_i^* while keeping $A_i < 0$. This change will decrease expected payment without changing the level of effort exerted. This is in contradiction to $(w_i^*, t_i^*, l_i^*)_{i=1,2}$ an optimal payments scheme. □

Claim 2. *In the optimal mechanism,* $l_i^* = 0$ *for* $i = 1, 2$.

Proof. Again by way of contradiction assume that for $i \in \{1, 2\}$, $l_i^* > 0$ and note that because $A_i \geq 0$ it must be the case that either $w_i^* > 0$ or $t_i^* > 0$ or both. Let l_i^* decrease together with either w_i^* or t_i^* or both, such that the effort level e^i does not change. This decreases the expected payment while preserving the exerted efforts, which is in contradiction to $(w_i^*, t_i^*, l_i^*)_{i=1,2}$, an optimal payments scheme. □

Claim 3. *In the optimal mechanism,* $t_i^* = 0$ *for* $i = 1, 2$.

Proof. If $e^1 = e^2$, the claim is correct, since $f_0'(0) = 0$. Assume that $e^1 > e^2$ and note that since $f_0'(e^1 - e^2) < 0$ it must be the case that $t_1^* = 0$ (decreasing t_1^* and w_1^* such that the total effort remains unchanged decreases expected payment) and it remains to show that $t_2^* = 0$. Assume, by contradiction, that $t_2^* > 0$ and consider decreasing t_2^* by ε and increasing w_2^* by

$$\frac{f_0'(e^2 - e^1)}{f_1'(e^2 - e^1)}\varepsilon.$$

The effect of this change on the right-hand side of equation (13) is

$$f_1'\left(e^2 - e^1\right)\left(w_2^* + \frac{f_0'(e^2 - e^1)}{f_1'(e^2 - e^1)}\varepsilon\right) + f_0'\left(e^2 - e^1\right)\left(t_2^* - \varepsilon\right)$$
$$-f_1'\left(e^2 - e^1\right)w_2^* - f_0'\left(e^2 - e^1\right)t_2^* = 0.$$

Therefore, this change does not affect the incentives of the agents to exert efforts. Now we will evaluate its effect on the expected payment. The expected payment after the change is

$$\begin{aligned}
& f_0\left(e^1 - e^2\right)\left(t_2^* - \varepsilon\right) + f_1\left(e^1 - e^2\right)w_1^* + f_2\left(e^1 - e^2\right)\left(w_2^* + \frac{f_0'(e^2 - e^1)}{f_1'(e^2 - e^1)}\varepsilon\right) \\
= \; & f_0\left(e^1 - e^2\right)t_2^* + f_1\left(e^1 - e^2\right)w_1^* + f_2\left(e^1 - e^2\right)w_2^* \\
& + \left[f_2\left(e^1 - e^2\right)\frac{f_0'(e^2 - e^1)}{f_1'(e^2 - e^1)} - f_0\left(e^1 - e^2\right)\right]\varepsilon.
\end{aligned}$$

We now use assumption **Ae** to conclude that the proposed change decreases the expected payment, contradiction. □

Claim 4. *In the optimal mechanism, $w_1^* = w_2^*$.*

Proof. Consider an effort level $E = e^1 + e^2$. If e^1 and e^2 are equilibrium effort levels, the payments w_i^*, $i = 1, 2$ are given by

$$w_1^* = \frac{c'(e^1)}{f_1'(e^1 - e^2)} \text{ and } w_2^* = \frac{c'(e^2)}{f_1'(e^2 - e^1)}.$$

Therefore, we can write the expected payment as

$$\frac{f_1(e^1 - e^2)}{f_1'(e^1 - e^2)}c'\left(e^1\right) - \frac{f_2(e^1 - e^2)}{f_2'(e^1 - e^2)}c'\left(e^2\right).$$

To establish the claim we now show that for any $E > 0$,

$$\left(\frac{E}{2}, \frac{E}{2}\right) = \arg\min_{e^1, e^2} \frac{f_1\left(e^1 - e^2\right)}{f_1'\left(e^1 - e^2\right)} c'\left(e^1\right) - \frac{f_2\left(e^1 - e^2\right)}{f_2'\left(e^1 - e^2\right)} c'\left(e^2\right)$$

$$s.t.$$

$$e^1 + e^2 = E.$$

Alternatively, inserting $e^2 = E - e^1$, we want to show that

$$\frac{E}{2} = \arg\min_{e^1} \frac{f_1\left(2e^1 - E\right)}{f_1'\left(2e^1 - E\right)} c'\left(e^1\right) - \frac{f_2\left(2e^1 - E\right)}{f_2'\left(2e^1 - E\right)} c'\left(E - e^1\right).$$

The derivative of the maximand with respect to e^1 is given by

$$\begin{aligned} &2\left(c'\left(e^1\right) - c'\left(E - e^1\right)\right) + \frac{f_1\left(2e^1 - E\right)}{f_1'\left(2e^1 - E\right)} c''\left(e^1\right) + \frac{f_2\left(2e^1 - E\right)}{f_2'\left(2e^1 - E\right)} c''\left(E - e^1\right) \\ &+ 2\frac{f_2''\left(2e^1 - E\right) f_2\left(2e^1 - E\right)}{\left(f_2'\left(2e^1 - E\right)\right)^2} c'\left(E - e^1\right) - 2\frac{f_1''\left(2e^1 - E\right) f_1\left(2e^1 - E\right)}{\left(f_1'\left(2e^1 - E\right)\right)^2} c'\left(e^1\right), \end{aligned}$$

which is indeed equal to 0 for $e^1 = \frac{E}{2}$. □

Second-Order Conditions

Here we derive the second-order condition for the optimality of a symmetric mechanism in a one-stage game.

The second derivative, with respect to e^1, is given by

$$\begin{aligned} &2\left(c''\left(e^1\right) - c''\left(E - e^1\right)\right) + \left(2 - 4\frac{f_1''\left(2e^1 - E\right) f_1\left(2e^1 - E\right)}{\left(f_1'\left(2e^1 - E\right)\right)^2}\right) c''\left(e^1\right) + \\ &\left(2 - 4\frac{f_2''\left(2e^1 - E\right) f_2\left(2e^1 - E\right)}{\left(f_2'\left(2e^1 - E\right)\right)^2}\right) c''\left(E - e^1\right) + \\ &\frac{f_1\left(2e^1 - E\right)}{f_1'\left(2e^1 - E\right)} c'''\left(e^1\right) - \frac{f_2\left(2e^1 - E\right)}{f_2'\left(2e^1 - E\right)} c'''\left(E - e^1\right) + \\ &4\frac{f_2'''\left(2e^1 - E\right) f_2\left(2e^1 - E\right) + f_2''\left(2e^1 - E\right) f_2'\left(2e^1 - E\right)}{\left(f_2'\left(2e^1 - E\right)\right)^2} c'\left(E - e^1\right) - \\ &8\frac{\left(f_2''\left(2e^1 - E\right)\right)^2 f_2\left(2e^1 - E\right)}{\left(f_2'\left(2e^1 - E\right)\right)^3} c'\left(E - e^1\right) - \\ &4\frac{f_1'''\left(2e^1 - E\right) f_1\left(2e^1 - E\right) + f_1''\left(2e^1 - E\right) f_1'\left(2e^1 - E\right)}{\left(f_1'\left(2e^1 - E\right)\right)^2} c'\left(e^1\right) + \\ &8\frac{\left(f_1''\left(2e^1 - E\right)\right)^2 f_1\left(2e^1 - E\right)}{\left(f_1'\left(2e^1 - E\right)\right)^3} c'\left(e^1\right). \end{aligned}$$

At $e^1 = \frac{E}{2}$ the above expression becomes

$$4\left(c''\left(\frac{E}{2}\right) - 2\frac{f_1''(0)}{f_1'(0)}c'\left(\frac{E}{2}\right)\right)\left(1 - 2\frac{f_1''(0)\,f_1(0)}{(f_1'(0))^2}\right)$$
$$+2\frac{f_1(0)}{f_1'(0)}c'''\left(\frac{E}{2}\right) - 8\frac{f_1'''(0)\,f_1(0)}{(f_1'(0))^2}c'\left(\frac{E}{2}\right).$$

If this expression is positive, then $e^1 = \frac{E}{2}$ is indeed a minimizer. If $f_1''(0)$ and $f_1'''(0)$ are relatively small and $c'''(\cdot) > 0$, then it is a minimizer.

7.4 E. Endogenous Prize, Two Stages

Before the proof of the main theorem is presented, we establish the following technical result.

Lemma 8. *Without loss of generality we can assume that in the optimal mechanism the first-order conditions (14), (15), (16), and (17) hold with equality.*

Proof. The proof is done by establishing four simple claims. □

Claim 5. *In an optimal mechanism,* $P_i(0,i) - P_i(0,j) \geq 0$.

Proof. Assume by contradiction that $P_i(0,i) - P_i(0,j) < 0$ and note that decreasing $P_i(0,j)$ will not affect incentives, but will decrease the expected payment, which is in contradiction to the mechanism being optimal.

Claim 6. *In an optimal mechanism,*

$$f_1'(e^F - e^L)\left(P_i(j,i) - P_i(j,j)\right) + f_0'(e^F - e^L)(P_i(j,0) - P_i(j,j)) \geq 0, \tag{37}$$

□

Proof. If Claim 6 where not true, then $e^F = 0$ and since $e^F - e^L \leq 0$, $f_0'(e^F - e^L) \geq 0$. Therefore, for (37) to be negative, it must be the case that $P_i(j,j) > 0$. Therefore, there exists $\varepsilon > 0$ such that decreasing $P_i(j,j)$ by ε will keep the right-hand side of (37) non-positive, will not change e^L and e^F, will increase $c'(e^1)$, and will decrease the expected payment, which is in contradiction to the mechanism being optimal. □

We conclude that the first-order condition for the follower holds with equality.

Claim 7. *We can assume without loss of generality that in the optimal mechanism,*

$$f_1'(e^L - e^F)\left(P_i(i,i) - P_i(i,j)\right) + f_0'(e^L - e^F)(P_i(i,0) - P_i(i,j)) \geq 0. \tag{38}$$

Proof. First, note that if the left-hand side of (38) is negative, then $e^L = 0$ and $f_0'(e^L - e^F) \geq 0$. Therefore, for the left-hand side of (38) to be negative it must be the case that $P_i(i,j) > 0$. Consider decreasing $P_i(i,j)$ by ε and increasing $P_i(i,i)$ by

$$\varepsilon \frac{f_2\left(-e^F\right)}{f_1\left(-e^F\right)}.$$

There exists $\varepsilon > 0$, such that the left-hand side of (38) stays negative. These changes do not change the expected payment. For $\delta = 0$ they do not change the first-period incentives; but for $\delta > 0$ they do increase first-period incentives. □

Claim 8. *We can assume without loss of generality that in the optimal mechanism, $D \geq 0$.*

Proof. Assume by way of contradiction that in the optimal mechanism $D < 0$. Since in this case $e^1 = 0$, it is optimal to set the payments at the second stage to be equal to the optimal payments in the one-stage game (regardless of the results in the first stage). Claims 5-7 imply that for D to be negative, it must be the case that $u^L - u^F < 0$. Consider the mechanism such that if in the first stage there are a winner and a loser, their prizes at the second stage are switched. That is, if in the original mechanism the prizes were $P_i(s_1, s_2)$, then in the new mechanism $P_i^*(i, s_2) = P_i(j, s_2)$ for any $s_2 \in \{0, 1, 2\}$. The expected payment and the total second-stage incentives remain unchanged, while first-period incentives increase (since $u^L - u^F > 0$). A contradiction. □

Theorem 6

Before proving the theorem, we will show some properties of the optimal mechanism.

Claim 9. *In an optimal mechanism, $P_i(j,j) = P_i(0,j) = P_i(0,0) = 0$.*

Proof. Assume that $P_i(j,j) > 0$. Then decreasing both $P_i(j,j)$ and either $P_i(j,i)$ or $P_i(j,0)$ or both, such that the right-hand side of the FOC (16) does not change, keeps e^F and e^L constant, decreases the expected payment, and increases e^1. Using a similar argument it can be shown that $P_i(0,j) = P_i(0,0) = 0$. □

Proof of Theorem 6. (i) Using Lemma 8, $P_i(j,j) = 0$ and the first-order conditions for the second stage, we have

$$\begin{aligned} P_i(i,i) - P_i(i,j) &= \frac{c'(e^L)}{f_1'(e^L - e^F)} - \frac{f_0'(e^L - e^F)}{f_1'(e^L - e^F)}(P_i(i,0) - P_i(i,j)) \\ P_i(j,i) &= \frac{c'(e^F)}{f_1'(e^F - e^L)} - \frac{f_0'(e^F - e^L)}{f_1'(e^F - e^L)} P_i(j,0). \end{aligned} \tag{39}$$

Moreover, first-order conditions for the first-period effort become

$$c'(e^1) = f_1'(0)\left(u^L - u^F\right) + \delta\left(f_1(0)\, c'(e^L) + f_0(0)\, c'(e^T) + f_2(0)\, c'(e^F)\right). \tag{40}$$

Plugging (39) into (18) yields

$$\begin{aligned} u^L - u^F = {} & c(e^F) - c(e^L) + \frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} c'(e^L) - \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} c'(e^F) \\ & + (P_i(i,0) - P_i(i,j)) \left[f_0(e^L - e^F) - \frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} f_0'(e^L - e^F) \right] \\ & - P_i(j,0) \left[f_0(e^F - e^L) - \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} f_0'(e^F - e^L) \right] + P_i(i,j). \end{aligned} \tag{41}$$

Notice that the expected costs can be written as

$$f_0(0)\, CT + (1 - f_0(0))\, CNT$$

where CT is the expected payment in case of a tie in first stage, while CNT is the expected payment if there is a leader in the first stage. Plugging (39) into (20) gives us

$$\begin{aligned} CNT = {} & \frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} c'(e^L) + \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} c'(e^F) \\ & + (P_i(i,0) - P_i(i,j)) \left[f_0(e^L - e^F) - \frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} f_0'(e^L - e^F) \right] \\ & + P_i(j,0) \left[f_0(e^F - e^L) - \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} f_0'(e^F - e^L) \right] + P_i(i,j). \end{aligned} \tag{42}$$

□

Claim 10. *In an optimal mechanism, $P_i(j,0) = 0$.*

Proof. First we shall show that

$$f_0(e^F - e^L) - \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} f_0'(e^F - e^L) \geq 0. \tag{43}$$

If in the optimal mechanism $e^F \geq e^L$, then from **Ai** it follows that $f_0'(e^F - e^L) \leq 0$, which, in turn, implies (43). If $e^L > e^F$, then (43) follows from **Ae**. We can now conclude that in the optimal mechanism, $P_i(j,0) = 0$. Otherwise (42) and (41) imply that decreasing $P_i(j,0)$ to 0 and changing $P_i(j,i)$, such that e^L and e^F remains the same, decreases the expected payment and increases the effort in the first stage. □

The next three claims complete the proof of part (i).

Claim 11. *If $e^F \geq e^L$, then increasing $e^L - e^F$, while keeping $e^L + e^F$ constant, increases*

$$c(e^F) - c(e^L) + \frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)}c'(e^L) - \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)}c'(e^F).$$

Proof. Plugging $e^F = E - e^L$ into the previous expression and deriving with respect to e^L gives

$$\begin{aligned}
&c'(e^L)\left(1 - 2\frac{f_1''(2e^L - E)f_1(2e^L - E)}{(f_1'(2e^L - E))^2}\right) + \\
&c'(E - e^L)\left(1 - 2\frac{f_1''(E - 2e^L)f_1(E - 2e^L)}{(f_1'(E - 2e^L))^2}\right) \\
&+\frac{f_1(2e^L - E)}{f_1'(2e^L - E)}c''(e^L) - \frac{f_2(2e^L - E)}{f_2'(2e^L - E)}c''(E - e^L).
\end{aligned}$$

Since $f_2'(s) < 0$, it is sufficient to show that the first two lines in the expression above are positive. Since $E - e^L \geq e^L$,

$$\begin{aligned}
& c'(e^L)\left(1 - 2\frac{f_1''(2e^L - E)f_1(2e^L - E)}{(f_1'(2e^L - E))^2}\right) + \\
& c'(E - e^L)\left(1 - 2\frac{f_1''(E - 2e^L)f_1(E - 2e^L)}{(f_1'(E - 2e^L))^2}\right) \\
> \quad & c'(e^L)\left(2 - 2\frac{f_1''(2e^L - E)f_1(2e^L - E)}{(f_1'(2e^L - E))^2} - 2\frac{f_1''(E - 2e^L)f_1(E - 2e^L)}{(f_1'(E - 2e^L))^2}\right) > 0
\end{aligned}$$

where the first inequality follows from (a) and the convexity of c, while the second inequality follows from (b). □

Claim 12. *If $c''' > 0$, then there is no mechanism where $e^F > e^L$.*

Proof. Assume that there exists an optimal mechanism in which $e^F > e^L$. Consider increasing $P_i(i,i)$ and decreasing $P_i(j,i)$ such that $e^L + e^F$ remains unchanged, but $e^L = e^F$ (that is, $P_i(i,i) - P_i(i,j) = P_i(j,i)$). Since the total second-stage efforts remain unchanged, we have to check the effect on both the first-period incentives and the expected payment. In addition, let us change $P_i(i,0)$ such that

$$(P_i(i,0) - P_i(i,j))\left[f_0(e^L - e^F) - \frac{f_0'(e^L - e^F)}{f_1'(e^L - e^F)}f_1(e^L - e^F)\right] \tag{44}$$

will remain unchanged. It is possible, since from Assumptions **Ae** and **Ai,** for $e^L < e^F$,

$$0 < f_0(e^L - e^F) - \frac{f_0'(e^L - e^F)}{f_1'(e^L - e^F)} f_1(e^L - e^F) < f_0(0).$$

Therefore, if in the original mechanism $P_i\,(i,0) - P_i\,(i,j) > 0$, then since $f_0'(0) = 0$, keeping (44) constant implies decreasing $P_i\,(i,0)$ while keeping $P_i\,(i,0) - P_i\,(i,j) > 0$, which is possible. If in the original mechanism $P_i\,(i,0) - P_i\,(i,j) < 0$, then we should increase $P_i\,(i,0)$, which is always possible. ¿From the result for one-stage mechanism we know that

$$\frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} c'(e^L) + \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} c'(e^F)$$

is minimized among all couples (e^L, e^F) with $e^L + e^F = E$ when $e^L = e^F$. Therefore, from (42) it follows that these changes decrease the expected payment. We shall now show that they increase the incentives at the first stage. Since from Claim 11 the changes increase $u^L - u^F$, it is enough to show that these changes increase $c'(e^L) + c'(e^F)$, which is the case if $c''' > 0$. $\square$

To complete the proof of part (i) we have to show that there is no optimal mechanism with $e^F = e^L$.

Claim 13. *There is no optimal mechanism with* $e^L = e^F$*.*

Proof. Assume by way of contradiction that there exists an optimal mechanism, such that $e^L = e^F$. Consider the following changes:

1. Consider setting $P_i(i,0)$ and $P_i(i,j)$ to

$$P_i^*(i,0) = P_i^*(i,j) = P_i(i,0) f_0(0) + P_i(i,j)\,(1 - f_0(0))$$

while $P_i(i,i)$ is adopted such that e^L remains the same as in the original mechanism, that is,

$$P_i(i,i) = \frac{c'(e^L)}{f_1'(e^L - e^F)} + P_i^*(i,j).$$

Notice that these changes imply that there is no change in

$$P_i(i,0)\left[f_0(e^L - e^F) - \frac{f_0'(e^L - e^F)}{f_1'(e^L - e^F)} f_1(e^L - e^F)\right] + \\ P_i(i,j)\left[1 - f_0(e^L - e^F) + \frac{f_0'(e^L - e^F)}{f_1'(e^L - e^F)} f_1(e^L - e^F)\right]. \tag{45}$$

Notice that from (42) and (41) it follows that the expected payment and the first-period equilibrium effort do not change.

2. Consider in addition increasing $P_i(i,i)$ and decreasing $P_i(j,i)$ such that e^L goes up, e^F goes down, and $e^L + e^F$ remains unchanged. Since from the result of one-stage mechanism we know that

$$\frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} c'(e^L) + \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} c'(e^F)$$

is minimized at $e^L = e^F$, then

$$\frac{\partial}{\partial e^L} \left(\frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} c'(e^L) + \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} c'(e^F) \right) \bigg|_{e^L + e^F = E} = 0. \tag{46}$$

From Claim 11 it follows that

$$\frac{\partial}{\partial e^L} \left(c(e^F) - c(e^L) + \frac{f_1(e^L - e^F)}{f_1'(e^L - e^F)} c'(e^L) - \frac{f_1(e^F - e^L)}{f_1'(e^F - e^L)} c'(e^F) \right) \bigg|_{e^L + e^F = E} > 0 \tag{47}$$

while

$$\frac{\partial}{\partial e^L} \left(f_1(0)\, c'(e^L) + f_2(0)\, c'(e^F) \right) \bigg|_{e^L + e^F = E} = 0. \tag{48}$$

Therefore, (47), (48), and (41) imply that the first-stage effort went up. This together with (46) and (42), implies that the proposed changes increase the principal's expected utility. A contradiction. $\square$

(ii) We are now ready to show that $B(i,i) > B(i,j) \geq B(i,0) \geq B(0,0)$.

Claim 14. *In an optimal mechanism, (1) $P_i(0,j) = 0$, (2) $P_i(0,0) = 0$, (3) $P_i(j,j) = 0$, and (4) $P_i(i,j) = P_i(i,0)$.*

Proof. (1) Suppose in an optimal mechanism $P_i(0,j) > 0$. Consider decreasing both $P_i(0,j)$ and $P_i(0,i)$, such that the difference between the two remains unchanged. These changes decrease the expected payment, but do not affect incentives. A contradiction.

(2) Suppose in an optimal mechanism $P_i(0,0) > 0$. Consider decreasing this payment. It does not affect the incentives, but it decreases the expected payment. A contradiction.

(3) Suppose in an optimal mechanism $P_i(j,j) > 0$. Consider decreasing both $P_i(j,j)$ and either $P_i(j,i)$ or $P_i(j,0)$ or both, such that the right hand-side of the FOC (16) does not change. These changes keep e^F and e^L constant, decrease the expected payment, and increase e^1. A contradiction.

(4) Suppose that in an optimal mechanism $P_i(i,j) \neq P_i(i,0)$. Denote P^* by

$$\begin{aligned} P^* \;=\; & P_i(i,0) \left[f_0(e^L - e^F) - \frac{f_0'(e^L - e^F)}{f_1'(e^L - e^F)} f_1(e^L - e^F) \right] + \\ & P_i(i,j) \left[1 - f_0(e^L - e^F) + \frac{f_0'(e^L - e^F)}{f_1'(e^L - e^F)} f_1(e^L - e^F) \right]. \end{aligned}$$

Setting $P_i(i,0) = P_i(i,j) = P^*$ and $P_i(i,i)$ such that e^L and e^F remain unchanged insures that the efforts and the expected payment remain unchanged. $\square$

Proof of part (ii). Claim 13 together with Claims (9), and (10) allow us to rewrite the first-order conditions for the second period as follows

$$\begin{aligned} c'(e^T) &= f_1'(0)P_i(0,i) \\ c'(e^L) &= f_1'(e^L - e^F)\left(P_i(i,i) - P_i(i,j)\right) \\ c'(e^F) &= f_1'(e^F - e^L)P_i(j,i). \end{aligned}$$

Since in an optimal mechanism $e^L > e^F$, it must be the case that $P_i(j,i) < P_i(i,i) - P_i(i,j)$. Rearranging the inequality and using the fact that $P_i(j,j) = 0$ implies that $B(i,i) > B(i,j)$. It follows from $P_i(j,0) = 0$, and $P_i(i,j) = P_i(i,0)$ that

$$B(i,j) = P_i(i,j) + P_i(j,i) \geq P_i(i,0) + P_i(j,0) = B(i,0)$$

where last inequality follows since $P_i(0,0) = 0$. $\square$

We next show by way of example that the ranking of $B(i,j)$ and $B(0,i)$ is ambiguous. The first example shows that in an optimal mechanism, $B(i,j) > B(0,i)$.
Example: Assume $\delta = 0$ and the winning probabilities are:

$$\begin{aligned} f_1(x) &= \frac{1}{2\sqrt{2\pi}} \int_{0.1-x}^{\infty} \exp(-\frac{y^2}{8})dy \\ f_0(x) &= \frac{1}{2\sqrt{2\pi}} \int_{-0.1-x}^{0.1-x} \exp(-\frac{y^2}{8})dy \end{aligned}$$

and the cost function is given by

$$c(e) = \frac{e^2}{15 - e}.$$

In the optimal payments of this case $P_i(i,i) = 12.546$, $P_i(i,0) = 0.1372$, $P_i(j,i) = 7.8214$, and $P_i(0,i) = 5.681$. Note that $B(i,j) = 0.1372 + 7.8214$, and $B(0,i) = 5.681$ and in particular $B(i,j) > B(0,i)$. $\triangleleft$

The next example shows the opposite ranking. That is, $B(i,j) < B(0,i)$.
Example: Assume again that $\delta = 0$ and that the winning probabilities are

$$\begin{aligned} f_1(x) &= \frac{1}{2\sqrt{2\pi}} \int_{0.1-x}^{\infty} \exp(-\frac{y^2}{8})dy \\ f_0(x) &= \frac{1}{2\sqrt{2\pi}} \int_{-0.1-x}^{0.1-x} \exp(-\frac{y^2}{8})dy \end{aligned}$$

and that the cost function is given by

$$c(e) = \frac{1}{3}\left(\exp\left(e\right) - e - 1\right).$$

In the optimal payments of this case $P_i(i,i) = 0.753$, $P_i\left(i,0\right) = 0$, $P_i(j,i) = 0.379$, and $P_i(0,i) = 0.41$. Note that $B(i,j) = 0.379$, $B(0,i) = 0.41$,and in particular $B(i,j) < B(0,i)$. $\triangleleft$

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