Voting with Interdependent Values: The Condorcet Winner

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Abstract

We generalize the standard, private values voting model with single-peaked preferences and incomplete information by introducing interdependent preferences. Our main results show how standard mechanisms that are outcome-equivalent and implement the Condorcet winner under complete information or under private values yield starkly different outcomes if values are interdependent. We also propose a new notion of Condorcet winner under incomplete information and interdependent preferences, and discuss its implementation. The new phenomena in this paper arise because different voting rules (including dynamic ones) induce different processes of information aggregation and learning.

1 Introduction

In this paper we generalize the standard, private values voting model with single-peaked preferences and several alternatives by introducing interdependent preferences: the peak of each agent is determined both by the agent's private information and by the information available to the other voters. Since others' signals are their own private information, each voter is here ex ante uncertain about her own preferred alternative. In particular, dynamic voting processes can reveal and aggregate information along the way since agents respond to new information about other voters by adjusting their voting strategy.

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It is well known that, on the domain of single-peaked preferences, the binary simple majority relation is transitive and its maximal element is the Condorcet winner (see Black [1948]): this is the preferred alternative of the median voter. On this domain, one can thus escape Arrow's aggregation impossibility result (Arrow, [1951]). Moreover, under a private values assumption, the direct mechanism that chooses the median peak is dominant-strategy incentive compatible (or strategy proof) and thus one escapes also Gibbard and Satterthwaite's implementation impossibility result (see Gibbard [1973] and Sattherthwaite [1975]). Dasgupta and Maskin [2020] offer a recent, powerful axiomatic justification of the Condorcet rule on any domain where it is applicable. Accordingly, we focus here on the Condorcet winner under interdependent preferences and its potential implementation via static and dynamic voting procedures.

Our main results show how standard mechanisms that are outcome-equivalent and implement the Condorcet winner under complete information or under private values yield starkly different outcomes if values are interdependent. We also propose a novel notion of Condorcet winner under incomplete information and interdependent preferences and show how it can be implemented. The new phenomena in this paper arise because different voting rules induce different processes of information aggregation and learning.

In our model, several privately informed agents have single-peaked preferences over several alternatives, and each agent's peak is determined by his/her own private signal and by the mean signal of the other voters. The weighted average formula for interdependent preferences is the simplest and most often assumed one, both in the behavioral literature (see for example DeGroot [1974] and Charness and Rabin [2001]) and in the theoretical one (e.g. Jehiel and Moldovanu [2001]).

We start with a basic model where signals are binary. In addition to two "extreme" positions on the "left" and on the "right" that correspond to the binary signals, we also consider compromise alternatives that lie in between.[1] The interdependence of preferences is what makes possible compromises salient in this model with binary signals.

With two private signals corresponding to the two extreme positions, agents cannot manipulate the intensity of their private information. We show that, in such a framework, the complete-information Condorcet winner can be robustly implemented (e.g., in ex-post Nash equilibrium) via a direct mechanism and also, more relevant for practical applications,

[1] These compromise alternatives' locations may be endogenous. For example, during March 2019 the UK parliament struggled to identify and elect a compromise deal between the "hard" Brexit demanded by a large faction of the Tories, and the "soft" version, closer in spirit to economically remaining in the EU, supported by Labour and other smaller parties.

via a procedure that resembles the *amendment* voting procedure used by English-speaking democracies, several Scandinavian countries and Switzerland. In the amendment procedure alternatives are considered two-by-two, and the majority winner advances to the next stage, as in an elimination tournament. While traditionally the order in which alternatives are put to vote in the amendment procedure is fixed in advance (and hence it is independent of the voting outcomes along the way), our mechanism requires that the two alternatives that are put to vote at each stage are the two most extreme ones according to the order of single-peakedness.[2] Moreover, the Yes-No tallies in each binary vote must be announced before the next-stage voting. The reason for the positive result is that the considered sequential procedure —that considers two alternatives at a time—allows bidirectional learning about the preferences of both "leftists" and "rightists".

We next show how alternative procedures that modify either the order in which alternatives are put to vote, or the information revealed along the way, may fail to robustly implement the complete-information Condorcet winner because the learning cannot be satisfactorily performed.

Another prominent scheme that fails to always implement here the Condorcet winner is the *successive* procedure used in most continental European parliaments (including the EU parliament). In this voting mechanism, alternatives are put to vote, one after another, until one of them gets a majority. The failure - due to the fact that learning about the preferences of others only proceeds in one direction - occurs even under the agenda where, at each stage, the considered alternative that is put to vote is one of the two most extreme ones.[3] Recall that, under incomplete information of the private values types, the successive procedure with such an agenda implements the Condorcet winner.

We next show that the complete-information Condorcet winner cannot be robustly implemented if agents obtain several signals, and thus can manipulate the magnitude of these. Moreover, we also show that the Condorcet winner cannot be implemented in such a model even in the weaker sense of the Bayes-Nash equilibrium.

As suggested by the above negative results, we define a new notion of *incomplete-information Condorcet winner:* this is the alternative that would win against any other one a simple majority vote conducted among the incompletely informed agents equipped

[2] Note that our procedure is a valid, standard amendment procedure if there are only three alternatives, e.g. the status quo, a proposed reform and an amendment to the reform.

[3] An example of such agenda formation is given by the long-standing practice of the German parliament and its Weimar precursor: "if several proposals are made to the same subject, then the first vote shall be on the farthest-reaching proposal. Decisive is the degree of deviation from status quo."

with interdependent preferences. Under single-peaked preferences of the private values type, the new notion coincides with the standard complete information Condorcet winner, but this is not necessarily the case in our model with interdependent preferences. We show that an incomplete information Condorcet winner always exists in our model, and that it is Bayes-Nash implemented via a direct median mechanism. While dynamic procedures generally fail to implement the incomplete-information Condorcet winner, we finally show that the amendment procedure (with any agenda!) implements it under the assumption that voters behave myopically. Since Bayes-Nash implementation requires a cardinal notion of utility we adopt in this last part a quadratic utility functional form, and we explain how the results can be generalized to other utilities.

The rest of the paper is organized as follows: In the next Subsection we review the related literature. In Section 2 we describe the basic social choice model with binary signals, the calculation of the complete-information Condorcet winner with interdependent preferences and its robust implementation via static and dynamic mechanisms. In Section 3 we enrich the model by allowing voters to have more than two signals and we first show that complete-information Condorcet winner cannot be implemented in Bayesian equilibrium. Next, we propose a new notion of incomplete-information Condorcet winner, prove its existence, and show how it can be implemented. Section 4 concludes.

1.1 Related Literature

A sizable literature on voting allows departures from the private values, incomplete information paradigm, but restricts attention to only two alternatives - see for example the many papers following the pioneering contribution of Feddersen and Pesendorfer [1997]. But, there are only a few papers that study voting models with more than two alternatives and with interdependent values (note that interdependence generalizes the more ubiquitous assumption of common values).[4]

Implementation with interdependent valuations is analyzed by Jehiel and Moldovanu [2001] and Jehiel et al. [2006] under the assumption that monetary transfers are feasible. Feng et al. [2022] focus on robust implementation with one dimensional signals and without

[4]Dekel and Piccione [2000] analyzed sequential voting with interdependent values in a model with only two alternatives: sequentiality is with respect to individual voting. They showed that, although the history of the first votes should intuitively affect the behavior of the later voters, equilibrium conditioning on pivotality leads voters to ignore the revealed history. Ali and Kartik [2012] displayed other equilibria where voters do take into account the observed history.

monetary transfers

Closest to our present paper, Gruener and Kiel [2004] and Rosar [2015] analyze static voting mechanisms in a setting where agents have quadratic, interdependent preferences, focusing on a comparison of the mean and the median mechanisms. Moldovanu and Shi [2013] analyze voting in a dynamic setting where multi-dimensional alternatives appear over time and where voters are only partially informed about some aspects of the alternative. Piketty [2000] studies a two-period voting model where a large number of agents care about the decisions taken at both stages. Voting at the first stage reveals information about preferences that is relevant at the second stage. Piketty concludes that electoral systems should be designed to facilitate efficient communication, e.g. by opting for two-round rather than one-round systems—this is congruent with the kind of multi-stage procedures observed in committees and legislatures and also discussed in this paper.

Following the pioneering work by Farquharson [1969], almost the entire literature on binary, sequential voting with several alternatives assumed that agents are completely informed about the preferences of others (see Miller [1977], McKelvey and Niemi [1978] and Moulin [1979], among others, for early important contributions). Under complete information, the associated extensive form games are amenable to analysis by backward induction: voters can, at each stage, foresee which alternative will be finally elected, essentially reducing each decision to a vote among two alternatives. If a simple majority is used at each stage, then, whenever it exists, a Condorcet winner is selected by sophisticated voters, independent of the particular structure of the binary voting tree, and independent of its agenda.[5]

An early analysis of strategic, sequential voting under incomplete information with private values is Ordeshook and Palfrey [1988]. They constructed Bayesian equilibria for an *amendment* procedure with three alternatives and with preference profiles that potentially lead to a Condorcet paradox. Gershkov, Moldovanu and Shi [2017] (GMS hereafter) analyzed voting by qualified majority in the successive procedure via a model where agents' preferences are single-peaked and follow the private values paradigm.[6] In their study, the order in which alternatives are put to vote follows the order defining single-peakedness (or its reverse). Kleiner and Moldovanu [2017] generalized the GMS results to the class of all sequential, binary procedures with a *convex* agenda. Recall that in a binary, sequential procedure each

[5]If a Condorcet winner does not exist, then a member of the Condorcet cycle is elected. The influence of agenda manipulations has been emphasized by Austen-Smith [1987] and, more recently by Barbera and Gerber [2017].

[6]Their focus was on finding the welfare maximizing procedure. This is achieved by varying the thresholds needed for the adoption of each alternative.

vote is taken by (possibly qualified) majority among two, not necessarily disjoint, subsets of alternatives. Convexity says that if two alternatives a and c belong to the left (right) subset at a given node, then any alternative b such that $a < b < c$ (in the ideological order governing single-peakedness) also belongs to the left (right) subset.

Under single-peaked, private values preferences, Kleiner and Moldovanu showed that sincere voting constitutes an ex post perfect equilibrium in any voting game derived from a sequential, binary voting tree with any convex agenda.[7] An important corollary is that, if simple majority is used at each stage of the voting tree, the associated equilibrium outcome is always the complete information Condorcet winner. Thus, all sequential binary voting trees with convex agendas and all information policies are equivalent under single-peaked, private values preferences, and this theory cannot discriminate among them.

Posner and Vermeulen [2016] argue that a more or less evenly split decision by several judges, or by a jury, may be logically incompatible with a conviction based on guilt "beyond reasonable doubt". They propose a dynamic voting procedure where members learn about the positions of others and adjust their opinion, and also argue that a formal procedure where the revealed numbers of supporters for each option speak for themselves is better than an informal, hard to quantify deliberation. On the empirical side, Chappel et al. [2004] studied the Federal Open Market Committee's detailed voting patterns on monetary policy, and test the hypothesis that the chairman's preferred policy is a weighted average of her own and the other members' signals – the same functional form as the one adopted here.[8,9] Martin and Vanberg [2014] empirically test several models of legislative compromise in coalition governments, and conclude that these tend to be positions that average opinions in coalitions rather than representing, say, the view of the median coalition member. Ezrow et al. [2011] conducted an analysis of political parties in 15 Western European democracies from 1973 to 2003 and showed that the larger, mainstream parties tend to adjust their positions on the Left-Right spectrum in response to shifts in the position of the mean voter, while being less sensitive to policy shifts of their own supporters. The opposite pattern holds for smaller, niche parties.

[7] In other words, voters cannot gain by manipulating their vote, regardless of their beliefs about others' preferences, and regardless of the information disclosure policy along the voting sequence. Under a mild refinement, this equilibrium is unique.

[8] There are twelve members, and the chairman's weight on his own signal is estimated to be between 0.15 and 0.20. Chappel et al. take their cue from an earlier study by Yohe [1966] who writes "...there is also no evidence to refute the view that the chairman adroitly detects the consensus of the committee, with which he persistently, in the interests of Systems harmony, aligns himself."

[9] They also estimate the opposite influence of the chairman on members.

2 The Voting Model with Two Signals

There are $2n+1$ voters who collectively choose one of k available alternatives. The finite number of alternatives is suitable for the discussion of sequential, binary voting procedures as used in most parliaments and committees, e.g., the amendment procedure. For some theoretical results about static mechanisms, we shall also consider the case of a continuum of alternatives.

We identify alternatives with their locations on a left-right ideological spectrum, and the set of locations is $X = \{x_1, ..., x_k\}$. The location of alternatives is ordered and normalized so that $-1 = x_1 < x_2 < ... < x_k = 1$. Before voting, each agent i, $i = 1, ..., 2n+1$, obtains a signal $s_i \in \{-1, 1\}$. We note here that the precise specification of probabilities and beliefs does not play a role whenever we use robust implementation notions such as the ex-post Nash equilibrium. Hence, we leave here the signal distribution unspecified.

Each voter, $i = 1, ..., 2n+1$, has an "ideal" location y_i for the elected alternative. Voter i's ideal point depends both on her own private signal s_i and also on the mean of all other voters' private signals s_j, $j \neq i$. Let $\gamma \in \left[\frac{1}{2n+1}, 1\right]$ denote the weight that voters with signal -1 and $+1$ assign to their own signal, respectively. The ideal location $y_i(s_i, s_{-i}) \in [-1, 1]$ for voter i is

$$y_i(s_i, s_{-i}) = \gamma s_i + \frac{1-\gamma}{2n} \sum_{j \neq i} s_j. \tag{1}$$

Thus, preferences are assumed here to be *interdependent*, and the weight γ on own signal s_i, captures the level of interdependence. A special case is $\gamma = 1$, which yields the private values case (no interdependence), while $\gamma = \frac{1}{2n+1}$ yields the pure common values case where, ex post, all voters share the same ideal point. Note that, in order to avoid a more complex, asymmetric model where voters with the same private signal s_i may have different weight γ_i, we assumed that the degree of interdependence is the same for all voters.

Interdependent preferences may arise due to different considerations. For example, voters may have other-regarding preferences and the parameter γ is then a measure of the voters' degree of altruism. In dynamic contexts (e.g., some political process), the interdependence may represent in reduced form the effects of a future interaction among the agents. Alternatively, one can interpret the interdependent preference as a result of a biased information aggregation. Voters try to learn some state but voter i puts more weight on their own signal rather than on other voters' signals.

The linear form of interdependent preferences is not critical for our analysis. What is important is that, when all signals are public information, voters' preferences are single-peaked. We thus assume that if alternative $x \in X$ is elected, the utility of voter i with ideal point y_i is given by $u(x, y_i)$ where $u(\cdot, y_i)$ is single-peaked at, and symmetric around y_i. In particular, any utility function $u(x, y_i)$ that is monotonically decreasing in the absolute value of the difference between y_i and x is feasible.

The linear form (1) and the symmetry assumption about $u(\cdot, y_i)$ allow us to compare voters' preferences over alternatives away from their ideal location y_i, without making explicit assumption on the functional form $u(\cdot, y_i)$. For example, common specifications in the Political Science literature are:

$$\begin{aligned} u(x, y_i) &= -(x - y_i)^2, \\ u(x, y_i) &= -\left|x - y_i\right|. \end{aligned}$$

Remark 1 *Given the linear structure (1) and the symmetry and single-peakedness of $u(\cdot, y_i)$, voter i's ex post ranking between two alternatives (x_i and x_j) only depends on their distance from voter i's ideal point y_i. Therefore, whenever we use robust implementation notions such as the ex-post Nash equilibrium, neither the signal distribution nor the cardinality of the utility function plays a role. In contrast, in our later extension with the weaker Bayesian implementation notion, both the signal distribution and the functional form of the utility may matter.*

2.1 The Complete Information Condorcet Winner

An alternative is the *complete information Condorcet winner* (Condorcet winner for short) if it wins in pair-wise simple-majority voting against any other alternative when all voters' signals are public information. For any given realization of signals, the assumed preferences are here single-peaked according to the left-right natural order $x_1, ..., x_k$ (or $x_k, ..., x_1$). Therefore, the complete information Condorcet winner always exists in our model.

An alternative x_l is the Condorcet winner if it is the alternative that is closest to the ideal point of voters whose signal is in the **majority**. Let n_{-1} denote the realized number of voters with signal -1 and $n_{+1} = 2n + 1 - n_{-1}$ denote the realized number of voters with signal $+1$. Voters with signal $+1$ form a majority if $n_{-1} \leq n$, and voters with signal -1 form a majority if $n_{-1} \geq n + 1$. Let y_m be the ideal point of voters in the majority. It follows from (1) that

$$y_m = \begin{cases} 1 - (1 - \gamma)\frac{n_{-1}}{n} & \text{if} \quad n_{-1} \leq n \\ -1 + (1 - \gamma)\frac{n+1}{n} & \text{if} \quad n_{-1} \geq n + 1 \end{cases}. \tag{2}$$

Therefore, alternative x_l is the Condorcet winner if y_m lies between the mid-point of x_{l-1} and x_l and the mid-point of x_l and x_{l+1}:

$$--\underset{x_{l-1}}{|}----\underbrace{|----\underset{x_l}{|}-------|}_{CW=x_l \text{ if } y_m \text{ lies in this interval}}\;-------\underset{x_{l+1}}{|}--$$

Formally, we define the Condorcet winner as[10]

$$CW = x_l \text{ if and only if } \frac{1}{2}(x_{l-1} + x_l) < y_m \leq \frac{1}{2}(x_l + x_{l+1}). \tag{3}$$

2.2 Direct Implementation of the Condorcet Winner

We first show that the social choice function selecting he Condorcet winner for any realization of signals is implementable in ex-post equilibrium. For the voting model with only two signals, this is an extension of Black's [1948] famous insights obtained for private values and for dominant strategy implementation.

Proposition 1 *Consider the direct mechanism $\Gamma^{CW} : \{-1, 1\}^{2n+1} \to X$ that chooses the Condorcet winner given by (3) for every profile of reports. Then, the strategy profile where each agent truthfully reports her signal is an ex-post Nash equilibrium.*

Proof. Consider the incentives of an arbitrary voter i with signal s_i, assuming that all the other agents report their signals truthfully. First, suppose that, among voters other than i, there are exactly n agents with signal $+1$ and n agents with signal -1. In this case, voter i is in the majority, and by definition (3), the Condorcet winner is voter i's most preferred alternative. Therefore, it is optimal for voter i to report truthfully under Γ^{CW}.

Next, suppose that, among voters other than i, the number of voters with signals $+1$ is greater than the number of voters with signals -1. If $s_i = +1$, voter i is in the majority and hence the Condorcet winner is again voter i's most preferred alternative. Thus, voter i has incentive to report his signal truthfully under mechanism Γ^{CW}. If $s_i = -1$, the ideal point of the majority when voter i reports $\hat{s}_i = +1$ is larger than the ideal point of the majority when voter i reports truthfully (due to interdependent preferences). Hence, voter i's misreporting, relative to truthful reporting, can only push the chosen alternative further away from voter i's ideal location.

A similar argument applies to the remaining case where, among voters other than i, the number of voters with $+1$ signals is smaller than the number of voters with -1 signals. ■

[10] The Condorcet winner is essentially unique. In the knife-edge case with $y_m = \frac{1}{2}(x_l + x_{l+1})$, both x_l and x_{l+1} are Condorcet winners. In this case, we let $CW = x_l$.

The above Proposition requires symmetry between the agents. It does generalize to the case where all agents who obtain signal -1 $(+1)$ use weights γ_{-1} (γ_{+1}), respectively. But, the implementation result may fail with more heterogeneity in these weights.

For example, suppose that the degree of interdependence can take three values $\{\gamma_{-1}, \gamma_0, \gamma_1\}$ with $\gamma_0 < \gamma_1$ and that the set of alternatives is given by

$$X = \left\{-1, -\frac{k-1}{k}, ..., 0, ..., \frac{k-1}{k}, 1\right\}$$

with sufficiently large k. Suppose that voters with signal -1 use weight γ_{-1}, but voters with signal $+1$ use the weights γ_0 and γ_1 with equal probability. Consider then voter i with signal $s_i = +1$ and $\gamma_i = \gamma_0$ under mechanism Γ^{CW}. When γ_0 is sufficiently close to $1/(2n+1)$ and voters with signal $+1$ have the majority, then voter i may have incentive to report $\hat{s}_i = -1$.

2.3 Indirect Implementation through Sequential, Binary Voting

The above direct mechanism requires the designer to know how voters' ideal points depend on the realization of signals. While it does implement the Condorcet winner, it is not a natural mechanism and it is not used in practice. Hence, we describe a natural indirect mechanism that always elects the Condorcet winner through binary, sequential voting. The proposed multi-stage sequential voting procedure is similar to the amendment procedure commonly used in Anglo-Saxon parliaments and many other committees, and will be called "iterated elimination of extreme alternatives (IEEA)."

Under the IEEA procedure, at every voting stage two extreme alternatives in the set of still available alternatives are put up for a vote by simple majority. The alternative that gets less support is eliminated from the set of the available alternatives. At the last stage, the last two remaining alternatives are paired up against each other and the alternative with majority support is selected.[11]

Formally, at the first stage, the set of the available alternatives is $X = \{x_1, ..., x_k\}$, and the two extreme alternatives, x_1 and x_k, are put up for vote. If alternative x_1 (or x_k) fails to get the majority support, it is eliminated from the set of available alternatives which then becomes $\{x_2, ..., x_k\}$ (or $\{x_1, ..., x_{k-1}\}$). In the second stage, there is a vote between x_2 and x_k (or between x_1 and x_{k-1}), as these are the extreme alternatives in the respective current set of remaining available alternatives. The process continues until the majority winner is chosen between the last two remaining alternatives. In each stage, the margin of the vote is revealed (or similarly the number of votes for every alternative is revealed).

[11] This is an example of a binary agenda that is convex in the terminology of Kleiner and Moldovanu [2017]. They prove that, in the private value setting, sincere voting is an ex post equilibrium under such an agenda.

Proposition 2 *The IEEA voting procedure has an ex-post Nash equilibrium where the complete information Condorcet winner is elected at each realization of signals.*

Proof. Consider the following candidate equilibrium strategy. At the first stage where alternatives x_1 and x_k are put up for vote, voters with signals -1 vote for x_1 while voters with signals $+1$ vote for x_k. From the second stage on, every agent votes in favor of the alternative that is closer to her ideal point (in case of a tie, voters are assumed without loss of generality to vote for the alternative with the lower index). If all voters follow this strategy, all information will be revealed after the first stage voting, and the continuation game from the second stage on becomes a game of complete information. Because from the second stage on every agent knows their own ideal point and ideal alternative, and because the IEEA procedure only eliminates alternatives in the intermediate stages and elects the winner only at the final stage, the Condorcet winner is always selected.

It remains to show that it is an ex post equilibrium for every voter to follow this strategy. To this end, consider the incentives of voter i, assuming that all the other voters follow this strategy. There are three cases to consider.

1. Among voters other than i, there are equal numbers of $+1$ and -1 signals. In this case, voter i is in the majority and the Condorcet winner is his most preferred alternative. Since the Condorcet winner is chosen in the candidate equilibrium, it is optimal for i to follow the recommended strategy.

2. Among voters other than i, there are more $+1$ signals than -1 signals. If voter i's signal is $+1$, voter i is in the majority and the Condorcet winner is his most preferred alternative. Hence, it is optimal for voter i to follow the above strategy. If voter i's signal is -1, the Condorcet winner is not voter i's most preferred alternative, but it remains optimal for voter i to follow the above strategy. To see this, note that deviating from the above strategy at any but the first stage will have no impact as there is a majority of agents with signal $+1$ who support the Condorcet winner. By deviating at the first stage and reporting $+1$ instead of -1, voter i increases the ideal point of the majority and hence (weakly) moves the chosen alternative further away from the ideal point of voter i. Hence voter i cannot gain from a deviation.

3. Among voters other than i, there are fewer $+1$ signals than -1 signals. This case is analogous to case 2. This completes the proof.

■

The IEEA procedure elects the Condorcet winner even if voters cannot communicate with each other prior to voting. This procedure has three notable properties. First, the most extreme alternatives are put up for vote first. Second, voters' private information is fully revealed in the first-stage voting. Third, the chosen alternative is not determined until the last stage. All three features are important for the ability of the IEEA procedure to always elect the Condorcet winner. We next illustrate the necessity of each property in the context of voting with three alternatives.

2.4 Departures from the IEEA Procedure

We consider below three departures from the IEEA procedure: each one violates one or more necessary properties for implementing the Condorcet winner.

Suppose that there are three alternatives: $X = \{x_1, x_2, x_3\}$ with $-1 = x_1 < x_2 < x_3 = 1$. The IEEA procedure requires that voters choose between x_1 and x_3 at the first stage, and that at the second stage after the margin of vote in the first stage is revealed, they choose between x_2 and the winner of the first stage. This procedure guarantees the selection of the Condorcet winner.

We first depart from the IEEA procedure by putting the moderate alternative x_2 up for vote in the first stage. Specifically, under this alternative voting procedure, voters choose between x_1 and x_2 at the first stage and at the second stage they choose between x_3 and the winner of the first stage.[12] We argue that this procedure may fail to select the Condorcet winner.

Lemma 1 *If $\gamma < \frac{1}{2}(1 + |x_2|)$ then the alternative procedure that puts x_1 and x_2 up for vote in the first stage does not always elect the Condorcet winner.*

Proof. Suppose by contradiction that there is a pure strategy profile that always selects the Condorcet winner. Let σ denote the corresponding profile of actions for the first stage. For each voter i, this yields a mapping $\sigma_i : \{-1, +1\} \to \{x_1, x_2\}$ where x_1 and x_2 in the profile of actions denote an action of voting in favor of x_1 and x_2, respectively.

Consider without loss of generality the case where $x_2 \leq 0$ so that $\gamma < \frac{1}{2}(1 - x_2)$. The other case with $\gamma < \frac{1}{2}(1 + x_2)$ can be proved analogously. If $n_{-1} = n + 1$, alternative x_2 is the Condorcet winner: the ideal point of voters with signal -1 is $-\gamma$ and it is closer to x_2 than to x_1 because $-\gamma > \frac{1}{2}(x_1 + x_2)$ is implied by $\gamma < \frac{1}{2}(1 - x_2)$.

[12] This is an example of a binary agenda that is not convex (Kleiner and Moldovanu [2017]).

Let $I = \{i : \sigma_i(-1) = x_1\}$ denote the set of voters who vote for x_1 if they have signal -1, and let $\#I$ denote the number of voters in set I. If $\#I \geq n+1$, consider the signal realization where $n_{-1} = n+1$ and all voters with signal -1 are drawn from set I. Then the Condorcet winner x_2 is eliminated in the first stage, a contradiction. If $\#I \leq n$, consider the signal realization where $n_{-1} = 2n+1$. Then the Condorcet winner is x_1 but it is eliminated in the first stage, a contradiction. Since there is no pure strategy profile that always selects the Condorcet winner, there is also no mixed strategy profile that always selects the Condorcet winner. ∎

To understand the necessity of the disclosure of winning margin, we consider a voting procedure that is identical to the IEEA procedure except that only the winner (but not the winning margin) is revealed after the first-stage voting. This procedure fails to implement the Condorcet winner because the latter finely depends on the signal realization while the voters do not obtain the information necessary for correct aggregation.

Lemma 2 *If* $\max\left\{\frac{1}{4}(1+x_2), 1-\frac{n}{2}(1+x_2)\right\} < \gamma < \frac{1}{2}(1-x_2)$ *or if* $\max\left\{\frac{1}{4}(1-x_2), 1-\frac{n}{2}(1-x_2)\right\}$ $< \gamma < \frac{1}{2}(1+x_2)$, *then the modified IEEA voting procedure that reveals only the winner does not always elect the Condorcet winner.*

Proof. Suppose, by contradiction, that there is a pure strategy profile that always selects the Condorcet winner. Let σ^1 and σ^2 denote the corresponding profile of actions for the first and second stage, respectively. Formally, $\sigma_i^1 : \{-1,+1\} \rightarrow \{x_1, x_3\}$ and $\sigma_i^2 : \{-1,+1\} \times \{x_1, x_3\} \rightarrow \{x_2, \neg x_2\}$ where x_2 in the range of σ_i^2 denotes a vote in favor of alternative x_2, while $\neg x_2$ denotes voting against it.

Consider first the case where $\max\left\{\frac{1}{4}(1+x_2), 1-\frac{n}{2}(1+x_2)\right\} < \gamma < \frac{1}{2}(1-x_2)$. Let n_{-1}^* be the minimal number of -1 voters such that x_1 is the Condorcet winner. As we argued in the proof of Lemma 1, $\gamma < \frac{1}{2}(1-x_2)$ implies that alternative x_2 is the Condorcet winner if $n_{-1} = n+1$, and hence $n_{-1}^* \geq n+2$. Moreover, given $\gamma > 1 - \frac{n}{2}(1+x_2)$, alternative x_1 is the Condorcet winner when $n_{-1} = 2n$ because

$$-1 + (1-\gamma)\frac{1}{n} < -1 + \frac{n}{2}(1+x_2)\frac{1}{n} = \frac{1}{2}(-1+x_2).$$

It follows that $n_{-1}^* \in [n+2, 2n]$. Finally, the last condition $\frac{1}{4}(1+x_2) < \gamma$ ensures that, if x_1 wins the first stage, conditional on being pivotal voters with signal $+1$ prefer x_2 to x_1 and hence will vote for x_2:

$$1 - (1-\gamma)\frac{n_{-1}}{n} \geq 1 - (1-\gamma)\frac{2n}{n} > \frac{1}{2}(-1+x_2)$$

We can classify all voters into 4 groups according to their choices of actions $\sigma_i^1(-1)$ and $\sigma_i^2(-1, x_1)$ when they obtain signal -1:

$$\begin{aligned}
I_{-1}^{11} &= \left\{i : \sigma_i^1(-1) = x_1, \sigma_i^2(-1, x_1) = \neg x_2 = x_1\right\} \\
I_{-1}^{12} &= \left\{i : \sigma_i^1(-1) = x_1, \sigma_i^2(-1, x_1) = x_2\right\} \\
I_{-1}^{31} &= \left\{i : \sigma_i^1(-1) = x_3, \sigma_i^2(-1, x_1) = \neg x_2 = x_1\right\} \\
I_{-1}^{32} &= \left\{i : \sigma_i^1(-1) = x_3, \sigma_i^2(-1, x_1) = x_2\right\}
\end{aligned}$$

We use $\#I_{-1}^{11}, \#I_{-1}^{12}, \#I_{-1}^{31}$ and $\#I_{-1}^{32}$ to denote the corresponding size of each group above. Two observations are immediate. First, for all profiles of signal realizations with $n_{-1} = n+1$, alternative x_2 is the Condorcet winner. To prevent x_1 from winning both stages, we must have

$$\#I_{-1}^{11} \leq n, \tag{4}$$

because otherwise we can draw all $n+1$ voters with signal -1 from the set I_{-1}^{11} to get x_1 elected. Second, for the profile of signal realizations with $n_{-1} = 2n+1$, alternative x_1 is the Condorcet winner. In order for x_1 to win both stages, we must have

$$\#I_{-1}^{11} + \#I_{-1}^{12} \geq n+1 \quad \text{and} \quad \#I_{-1}^{11} + \#I_{-1}^{31} \geq n+1. \tag{5}$$

It follows from (4) and (5) that all three sets, I_{-1}^{11}, I_{-1}^{12}, and I_{-1}^{31}, are non-empty.

Now we argue that the set I_{-1}^{32} must be empty. Suppose not. Consider a profile A of signal realizations such that $n_{-1} = n_{-1}^*$ where at least one voter with signal -1 is drawn from I_{-1}^{32}. Consider another profile A' of signal realizations that is identical to profile A except that one voter drawn from I_{-1}^{32} with signal -1 is replaced by a voter with signal $+1$ who votes for x_3 in stage one and for x_2 in stage two if x_1 wins stage one.[13] By construction, the two profiles yield the same vote patterns at both stages, and hence will elect the same alternative. But, x_1 is the Condorcet winner under profile A while x_2 is the Condorcet winner under profile A', yielding a contradiction. Therefore, the set I_{-1}^{32} must be empty and we have

$$\#I_{-1}^{11} + \#I_{-1}^{12} + \#I_{-1}^{31} = 2n+1. \tag{6}$$

Consider now a profile B of signal realizations such that $n_{-1} = n_{-1}^*$, where m^{11} voters, $1 \leq m^{11} \leq \#I_{-1}^{11}$, with signal -1 are drawn from I_{-1}^{11}, m^{12} voters, $1 \leq m^{12} \leq \#I_{-1}^{12}$, with

[13] Note that x_3 is the Condorcet winner for signal realizations where $n_{-1} = 0$. In order to elect x_3 in the first stage, the set of voters with equilibrium strategy $\sigma_i^1(+1) = x_3$ must be non-empty. Moreover, given that $\gamma > \frac{1}{4}(1 + x_2)$, all voters with signal $+1$ vote for x_2 in stage two if x_1 wins stage one. Therefore, the replacement in the above construction is feasible.

signal -1 are drawn from I_{-1}^{12}, and $m^{31} = n_{-1}^{*} - m^{11} - m^{12}$ voters, $0 \leq m^{31} < \#I_{-1}^{31}$, with signal -1 are drawn from I_{-1}^{31}. Profile B is feasible because $n_{-1}^{*} \in [n+2, 2n]$. We first argue that there must be exactly $n+1$ voters who vote in favor of alternative x_1 at stage one. Suppose instead that x_1 gathers at least $n+2$ votes in stage one. Consider profile B' that is identical to B except that one voter with signal -1 who is drawn from I_{-1}^{12} is replaced by a voter with signal $+1$ who will vote for x_3 in stage one and vote for x_2 in stage two if x_1 wins stage one.[14] By construction, x_1 should also be elected with profile B', but x_2 is the Condorcet winner for profile B', yielding a contradiction. Therefore, under profile B, alternative x_1 gathers exactly $n+1$ votes at stage one. Finally, consider profile B'' with voter composition given by

$$\begin{aligned} \hat{m}^{11} &= m^{11} - 1 \\ \hat{m}^{12} &= m^{12} \\ \hat{m}^{31} &= m^{31} + 1 \end{aligned}$$

Then under profile B'', $n_{-1} = n_{-1}^{*}$ and hence x_1 is the Condorcet winner, but x_1 loses in stage one, yielding a contradiction. This completes the proof for the first case. The other case with $\max\left\{\frac{1}{4}(1-x_2), 1-\frac{n}{2}(1-x_2)\right\} < \gamma < \frac{1}{2}(1+x_2)$ is symmetric, and can be proved analogously. ■

The third alternative voting procedure we consider is the successive voting procedure that is commonly used in continental European parliaments. Alternatives are ordered according to an agenda, say $[x_1, \{x_2, x_3\}]$. With this agenda, voters first decide by simple majority to accept, or to reject alternative x_1. If x_1 is accepted, voting ends. Otherwise, voters decide whether to accept alternative x_2. Alternative x_2 is accepted if it has majority support and x_3 is accepted otherwise. We assume that the margin of the voting in the first stage is fully revealed.

Lemma 3 *If $\gamma < \frac{1}{2}(1+|x_2|)$ then the successive voting procedure with agenda $[x_1, \{x_2, x_3\}]$ does not always elect the Condorcet winner.*

Proof. To obtain a contradiction, suppose there is a pure strategy profile that always selects the Condorcet winner, and let σ denote the corresponding profile of actions for the first stage. For each voter i this yields a mapping $\sigma_i : \{-1, +1\} \to \{x_1, \neg x_1\}$ where x_1 in the profile of actions denotes an action of voting in favor of x_1, while $\neg x_1$ denotes voting against x_1.

[14] As argued in the previous footnote, there always exists a voter with a strategy such that, with signal $+1$, he votes for x_3 in stage one and votes for x_2 in stage two if x_1 wins stage one.

Consider again without loss the case where $x_2 \leq 0$, so that $\gamma < \frac{1}{2}(1-x_2)$. The other case with $\gamma < \frac{1}{2}(1+x_2)$ is analogous. Again, if $\gamma < \frac{1}{2}(1-x_2)$ and $n_{-1} = n+1$, alternative x_2 is the Condorcet winner.

Let $I = \{i : \sigma_i(-1) = x_1\}$ denote the set of voters who vote for x_1 if they have signal -1. If $\#I \geq n+1$, consider the signal realization where all voters in I get signal $+1$ and where $n_{-1} = n+1$. Then all voters with signal -1 would vote for x_1 and hence x_1 is elected, but x_1 is not the Condorcet winner, a contradiction. If $\#I \leq n$, then x_1 will not be selected even if all voters have signal -1, in which case x_1 is the Condorcet winner, a contradiction. ∎

Why does the successive voting procedure fail to select the Condorcet winner? First, information may not be fully revealed in the first vote on x_1. In particular, if $\gamma < \frac{1}{2}(1-x_2)$, voters may unanimously reject x_1 even though x_1 may be the Condorcet winner. Second, the winner may be chosen in the first-stage voting. In particular, if $\gamma < \frac{1}{2}(1-x_2)$, alternative x_1 may be chosen even though it is not the Condorcet winner.

An alternative interpretation of the above Lemma is that, in order to implement the Condorcet winner via the successive voting procedure, it is necessary to add another stage of information revelation such that the successive procedure is conducted under complete information[15]. For example, if we add a preliminary stage in which voters vote whether to have agenda $[x_1, \{x_2, x_3\}]$ or agenda $[x_3, \{x_2, x_1\}]$, then there exists an equilibrium with information revelation at the first stage where the Condorcet winner will be elected.[16] We note that such preliminary votes on the agenda itself are sometimes conducted in reality (see for example, Kleiner and Moldovanu [2020] for a case from the Weimar republic).

3 The Voting Model with a Rich Signal Space

Let us now consider a richer set of signals, and assume that $s_i \in [-1, 1]$. The model is otherwise the same as above: the ideal point $y_i(s_i, s_{-i})$ of of voter i is

$$y_i(s_i, s_{-i}) = \gamma s_i + \frac{1-\gamma}{2n} \sum_{j \neq i} s_j.$$

The complete information Condorcet winner is, again, the alternative that is preferred by the median voter or, equivalently, the alternative that is closest to the median voter's ideal point.

[15] Before participation in the successive procedure agents may also be involved in some cheap-talk interaction that reveals their private information. Then, there exists an equilibrium in successive procedure with a cheap talk stage that will elect the Condorcet winner.

[16] We are grateful to anonymous referee for suggesting this procedure.

If there are three or more signals, voters can manipulate the magnitude of their signals. This incentive is sometimes so strong that the full-information Condorcet winner cannot be robustly implemented.

Proposition 3 *If $\gamma \in \left(\frac{1}{2n+1}, 1\right)$ and $k \geq 3$ then there is no ex-post incentive compatible mechanism that always selects the full-information Condorcet winner.*

Proof. Suppose by contradiction that there is a mechanism Γ^{CW} that always selects the Condorcet winner in ex-post equilibrium. Consider two profiles of signal realizations $(s_1, ..., s_i, ..., s_{2n+1})$ and $(s_1, ..., s_i', ..., s_{2n+1})$ that have following properties: (1) they differ only in the signal of voter i and have the same median voter $m \neq i$ with signal s_m; (2) signal s_i is interior with $-1 \leq s_m < s_i < s_i' \leq 1$; (3) given signal realizations $(s_1, ..., s_j, ..., s_{2n+1})$, the median voter m is almost indifferent between alternatives x_j and x_{j+1}, but slightly prefers x_j and hence the Condorcet winner is x_j; (4) given signal realizations $(s_1, ..., s_j', ..., s_{2n+1})$, voter m slightly prefers x_{j+1} and hence the Condorcet winner is x_{j+1}; (5) voter i prefers x_{j+1} to x_j under both signal realizations. Since $\gamma \in \left(\frac{1}{2n+1}, 1\right)$, since there are at least two alternatives, and since signals are assumed to be continuous, such profiles exist. By assumption, Γ^{CW} always selects the Condorcet winner, and hence $\Gamma^{CW}(s_1, ..., s_i, ..., s_{2n+1}) = x_k$ while $\Gamma^{CW}(s_1, ..., s_i', ..., s_{2n+1}) = x_{k+1}$. But given signal realization $(s_1, ..., s_i, ..., s_{2n+1})$, voter i has an incentive to misreport his signal to be s_i'. This yields a contradiction. ■

We note that in the present framework with a continuum of signals, a much stronger impossibility result holds: only constant social choice functions are ex-post implementable - see Feng et al [2022]. These authors also discuss why ex-post implementation is more permissive in models with discrete (e.g., binary) signals, a phenomenon we also observed in our model with binary signals analyzed above.

3.1 Bayesian Implementation with Quadratic Utilities

The previous impossibility result suggests that we may need to relax our equilibrium concept to Bayesian implementation. For this relaxation, we need both assumptions about the distribution of signals and a cardinal specification of utilities.

We assume that each voter's signal is drawn I.I.D. from the interval $[-1, 1]$ according to a bounded density $f > 0$, and that preferences are quadratic: $u(x, y_i) = -(x - y_i)^2$.

Finally, for additional tractability in this cardinal framework (e.g., in order to uses calculus), we assume for the sequel that the set of feasible alternatives is the entire interval $X = [-1, 1]$.

Proposition 4 *If* $\gamma \in \left(\frac{1}{2n+1}, 1\right)$ *and* $k \geq 3$ *then there is no Bayesian incentive compatible mechanism that always selects the full-information Condorcet winner.*

Proof. Without loss of generality, consider a direct mechanism that selects the Condorcet winner, the alternative which is the peak of the median voter under the assumption of truthful reports. Let us look at agent 1's incentives, assuming that all other voters report their types truthfully. Consider then the utility of voter 1 who observes signal s_1, but reports signal s_1'. Denote by s_l the n-th highest report of all agents but 1, and by s_h the $(n+1)$-th highest (or n-th lowest) report of all agents but 1. Denote by G the marginal distribution of s_l (with density g) and by Q the marginal distribution of s_h (density q). If $s_1' < s_l$, then the mechanism implements the alternative that is the peak of agent l (under truthful reports of all agents but 1 and report s_1' of voter 1). If $s_1' > s_h$, then the mechanism implements the alternative which is the peak of agent h (under truthful reports of all agents but 1 and report s_1' of voter 1). If $s_l < s_1' < s_h$ then the mechanism implements the alternative which is the peak of agent 1 with reports $(s_1', s_2, ..., s_n)$ that is, it is the bliss point of agent 1 if he would get signal s_1' and all others report truthfully. Observe that in the first case ($s_1' < s_l$) the difference between the true peak of agent 1 and the implemented alternative is

$$\gamma\left(s_1 - s_l\right) + \frac{1-\gamma}{2n}(s_l - s_1'),$$

in the second case ($s_1' > s_h$) the difference between the true peak of agent 1 and the implemented alternative is

$$\gamma\left(s_1 - s_h\right) + \frac{1-\gamma}{2n}(s_h - s_1'),$$

and in third case ($s_l < s_1' < s_h$) the difference is

$$\gamma\left(s_1 - s_1'\right).$$

Hence the expected utility of voter 1 with signal s_1 and report s_1' (assuming truthful reports of all other agents) is given by

$$\begin{aligned} &\int_{s_1'}^{1} -\left(\gamma\left(s_1 - s_l\right) + \frac{1-\gamma}{2n}(s_l - s_1')\right)^2 g\left(s_l\right) ds_l \\ &+ \int_{-1}^{s_1'} -\left(\gamma\left(s_1 - s_h\right) + \frac{1-\gamma}{2n}(s_h - s_1')\right)^2 q\left(s_h\right) ds_h \\ &- \left(\gamma\left(s_1 - s_1'\right)\right)^2 \Pr\left(s_l < s_1' < s_h\right). \end{aligned}$$

If there exists a Bayesian incentive compatible mechanism that implements the Condorcet winner, then the last expression should be maximized at $s_1' = s_1$. Taking the first-order condition with respect to s_1' and setting it to zero yields:

$$\begin{aligned}
0 &= \left(\gamma\left(s_1 - s_1'\right)\right)^2 g\left(s_1'\right) + \int_{s_1'}^{1} \frac{1-\gamma}{n}\left(\gamma\left(s_1 - s_l\right) + \frac{1-\gamma}{2n}(s_l - s_1')\right) g\left(s_l\right) ds_l \\
&\quad - \left(\gamma\left(s_1 - s_1'\right)\right)^2 h\left(s_1'\right) + \int_{-1}^{s_1'} \frac{1-\gamma}{n}\left(\gamma\left(s_1 - s_h\right) + \frac{1-\gamma}{2n}(s_h - s_1')\right) h\left(s_h\right) ds_h \\
&\quad +2\left(\gamma\left(s_1 - s_1'\right)\right) \Pr\left(s_l < s_1' < s_h\right) - \left(\gamma\left(s_1 - s_1'\right)\right)^2 \frac{\partial}{\partial s_1'} \Pr\left(s_l < s_1' < s_h\right).
\end{aligned}$$

The above equality must hold for $s_1' = s_1$, and this yields

$$\begin{aligned}
0 &= \int_{s_1}^{1} \frac{1-\gamma}{n}\left(\gamma\left(s_1 - s_l\right) + \frac{1-\gamma}{2n}(s_l - s_1)\right) g\left(s_l\right) ds_l \\
&\quad + \int_{-1}^{s_1} \frac{1-\gamma}{n}\left(\gamma\left(s_1 - s_h\right) + \frac{1-\gamma}{2n}(s_h - s_1)\right) q\left(s_h\right) ds_h \\
&= \frac{1-\gamma}{n}\left(\frac{1-\gamma}{2n} - \gamma\right) \int_{s_1}^{1} (s_l - s_1) g\left(s_l\right) ds_l \\
&\quad + \frac{1-\gamma}{n}\left(\frac{1-\gamma}{2n} - \gamma\right) \int_{-1}^{s_1} (s_h - s_1) q\left(s_h\right) ds_h.
\end{aligned}$$

Since $\gamma \in \left(\frac{1}{2n+1}, 1\right)$, this is equivalent to

$$0 = \int_{s_1}^{1} (s_l - s_1) g\left(s_l\right) ds_l + \int_{-1}^{s_1} (s_h - s_1) h\left(s_h\right) ds_h$$

For the mechanism to be BIC, the last equality must hold for any s_1. Taking the derivative on both sides of the last equality with respect to s_1, we finally obtain:

$$0 = -\int_{s_1}^{1} g\left(s_l\right) ds_l - \int_{-1}^{s_1} q\left(s_h\right) ds_h.$$

The above yields a contradiction since $g\left(s_l\right) > 0$ and $q\left(s_h\right) > 0$. ■

3.2 The Incomplete-Information Condorcet Winner

The above impossibility result under the weaker equilibrium concept suggests a new, modified notion of Condorcet winner that is consistent with the presence of incomplete information, and that can be potentially implemented in Bayesian equilibrium. We first need the following Lemma about simple majority voting among two alternatives.

Lemma 4 *Consider voting by simply majority among two alternatives x and y in $X = [-1, 1]$, and assume without loss of generality that $x < y$. This game has a unique symmetric Bayes-Nash equilibrium in undominated strategies (*equilibrium *for short). In this equilibrium, there is a cutoff $c \in [-1, 1]$ such that voter i votes for x if $s_i < c$ and votes for y if $s_i > c$. When interior, this cutoff is determined by the equation*

$$\gamma c + \frac{1-\gamma}{2}\left[\mathbb{E}[s|s<c] + \mathbb{E}[s|s>c]\right] = \frac{x+y}{2}$$

Proof. When determining her optimal strategy, a voter only needs to consider the event where she is pivotal. Assuming that all other voters use a strategy with cutoff c as above, and conditioning on being pivotal, the preferred alternative (i.e., the peak) of voter i with signal s_i (within the entire set $X = [-1, 1]$!) is given here by

$$\gamma s_i + \frac{1-\gamma}{2} \left[\mathbb{E}[s|s < c] + \mathbb{E}[s|s > c]\right]$$

Moreover, preferences are single-peaked around the above ideal point.

Note that the expression for the peak is strictly increasing in s_i, and hence the equilibrium strategy of voter i is also determined by a certain cut-off such that she votes for x for signals below the cutoff and votes for y for signals above the cutoff. In a symmetric cutoff equilibrium all voters use a strategy with the same cutoff, and, when interior, the equilibrium cutoff is determined by the fact that a voter who has a signal equal to the cutoff must be indifferent between voting in favor of x or in favor of y, i.e. her preferred alternative must be midway between x and y :

$$\gamma c + \frac{1-\gamma}{2} \left[\mathbb{E}[s|s < c] + \mathbb{E}[s|s > c]\right] = \frac{x+y}{2}$$

Existence follows by continuity, and uniqueness follows because the left hand side is strictly increasing in c. The fact that all symmetric Bayes-Nash equilibria in undominated strategies have the above cutoff form follows here analogously to Proposition 1 in Feddersen and Pesendorfer [1997]. ■

We are now ready to define our notion of Condorcet winner:

Definition 1 **a** *We say that an alternative* x wins against alternative y under incomplete information at signals $(s_1, ..s_{2n+1})$ *if in the equilibrium of the simple majority vote between* x *and* y*,* x *is elected when the realization of signals is* $(s_1, ..s_{2n+1})$.

b *An alternative* $x_{CW} = x_{CW}(s_1, ..s_{2n+1})$ *is an* incomplete information Condorcet winner at $(s_1, ..s_{2n+1})$ *if, for any other alternative* $x \in X$*,* x_{CW} *wins against* x *under incomplete information at* $(s_1, ..s_{2n+1})$.

The above notion mimics the logic of the classical one under complete information, and coincides with it in that case and in the incomplete information case with private values. But, the two notions need not be the same when values are interdependent. For example, we show below that the incomplete information Condorcet winner can be Bayes-Nash implemented, contrasting the impossibility result obtained in the previous section.

Note that the transitivity of the binary relation defined above is sufficient but not necessary for the existence of a maximal element, i.e. a Condorcet winner.[17,18]

Lemma 5 **a** *For any fixed realization of signals* $(s_1, ..s_{2n+1})$*, the binary relation among alternatives "*x wins against y under incomplete information at $(s_1, ..s_{2n+1})$*" is transitive.*

b *An incomplete information Condorcet winner exists and is unique for any realization of signals. If* s_i *is a median signal when realized signals are* $(s_1, ..s_{2n+1})$ *then*

$$\gamma s_i + \frac{1-\gamma}{2} \left[\mathbb{E}[s|s < s_i] + \mathbb{E}[s|s > s_i]\right]$$

is the incomplete information Condorcet winner at this realization.

Proof. Let us start with a direct proof of point **b)**: Fix a profile of signals, and let s_i be the median one, i.e., n voters have a signals weakly above (below, respectively) s_i. Define

$$x_{CW} := \gamma s_i + \frac{1-\gamma}{2} \left[\mathbb{E}[s|s < s_i] + \mathbb{E}[s|s > s_i]\right],$$

and consider a simple vote between x_{CW} and $x \neq x_{CW}$ according to simultaneous, simple majority voting.

Assume first that $x > x_{CW}$. By the above Lemma the equilibrium is such that all voters with signal above some cutoff c vote for x, and all voters with signal below c vote for x_{CW}. Conditional on i being pivotal and on other voters using cutoff strategies with cutoff c, the expected ideal point for voter i with signal s_i is

$$\gamma s_i + \frac{1-\gamma}{2} \left[\mathbb{E}[s|s < c] + \mathbb{E}[s|s > c]\right].$$

For $c = s_i$, this is precisely x_{CW}, and thus voter i would strictly prefer x_{CW} over x. Since the expected ideal point is increasing in c, we obtain that $c > s_i$ must hold. Therefore, voter i and all voters with lower signals vote for x_{CW}. Since voter i had, by assumption, the median signal, alternative x_{CW} is elected. Hence, any alternative x such that $x > x_{CW}$ loses against x_{CW} in any equilibrium.

The arguments for $x < x_{CW}$ are symmetric, and it follows that x_{CW} is the unique incomplete information Condorcet winner at the given profile.

[17] An example is the space of preferences that are single-peaked on a tree. The Condorcet winner always exist, but the majority relation is not necessarily transitive.

[18] If the space of alternatives is infinite, then transitivity needs to be complemented by a compactness assumption in order to ensure the existence of a maximal element, i.e. the Condorcet winner.

We now turn to transitivity, point **a)**. Fix a realization of signals $(s_1, ..s_{2n+1})$ and assume without loss of generality that $x < y$, and that x wins against y under incomplete information at this realization. Note that the cutoff in the corresponding equilibrium of simple majority voting between x and y is increasing in both x and y.

Since by assumption x wins against y, we know that there are at least $n+1$ signals below c_{xy}. Assume now that y wins against z. We need to show that x wins against z. If $z > y$ then $c_{xz} > c_{xy}$, and thus there are obviously at least $n+1$ signals below c_{xz}, and x wins against z.

Assume then that $z < y$. Hence there are at least $n+1$ signals above c_{zy}. If $x < z < y$ then, by monotonicity, the configuration of cutoffs must be $c_{xz} < c_{xy} < c_{zy}$. But then we obtain that there are at least $n+1$ signals below c_{xy} and at least $n+1$ signals above c_{zy}, a contradiction. So here it cannot be the case that y wins against z.

The only remaining case is $z < x < y$, in which case the configuration of cutoffs must be $c_{zx} < c_{zy} < c_{xy}$. Since there are at least $n+1$ signals above c_{zy}, there are also at least $n+1$ signals above c_{zx} , and hence x wins against z as desired. ■

Remark 2 *The analysis in the above Lemma was based on quadratic utilities. But, the existence of an incomplete-information Condorcet winner generalizes to any utility function such that the majority voting among any two alternatives has a cutoff equilibrium (see general conditions for this to hold in Feddersen and Pesendorfer [1997]) and such that the respective cutoff is monotonically increasing in the location of both alternatives. Then, we can apply transitivity - that only hinges on the ordering of cutoffs - to yield the desired existence.*

The definition of the Condorcet winner under incomplete information is based on a collection of completely separate, binary votes. It is not a priori clear that this alternative be implemented also in the full-fledged strategic situation where the agents are aware of and select among several alternatives.

Proposition 5 *The direct revelation mechanism that always selects the expected peak of the agent with the median signal is Bayesian incentive compatible and implements the incomplete information Condorcet winner.*

Proof. This is essentially Proposition 1 in Gruener and Kiel [2004]. They considered the indirect mechanism where agents are asked to report their peaks and where the designer selects the median peak. These authors showed that the equilibrium strategy of an agent with signal s_i is to report the peak

$$\gamma s_i + \frac{1-\gamma}{2} \left[\mathbb{E}[s|s < s_i] + \mathbb{E}[s|s > s_i] \right]$$

Our result follows then by the revelation principle, and by our above definition of the incomplete-information Condorcet winner. ■

Contrasting the above direct implementation and the equivalence among various procedures under complete information or private values, sequential binary voting procedures do not necessarily implement the incomplete information Condorcet winner under interdependent values. The reason is that, as shown above, in sequential voting games, sophisticated agents dynamically learn about their preferred alternative from the respective announced results of previous votes. This gradual learning process is not reflected in the static definition of the incomplete information Condorcet winner.

Along strategic sophistication, another important and well-studied behavioral assumption in the voting literature is "naive voting". One facet of naivete is myopic voting. Assume then that agents behave naively in the sense that, at each stage of a sequential, binary voting procedure, they ignore both the already revealed information and the consequence of today's outcome on future play. We obtain:

Proposition 6 *Assume that agents vote myopically and that there is a finite number of alternatives. Then, for any realization of signals, the amendment procedure (with any agenda !) implements the incomplete information Condorcet winner .*

Proof. Assume that $x = x(s_1, s_2, ..s_n)$ is the elected alternative in an amendment procedure given signals $(s_1, s_2, ..s_n)$, and look at any other alternative $y = y_0 \neq x$. Consider alternative y_1 that eliminated y_0 in a direct vote among myopic voters. This is the same as saying that y_1 wins against $y_0 = y$ under incomplete information. Such an alternative must exist because in an amendment procedure all alternatives are put to vote at some stage, and because y_0 was not ultimately elected at this signal realization. If $y_1 = x$ then obviously x wins against y_0 under incomplete information. If $y_1 \neq x$, then consider alternative y_2 that directly eliminated y_1: this must exist by the same argument as above. Since the number of alternative is finite, and since x is ultimately elected, we can construct a chain such that $y = y_0, y_1, .., y_l = x$ such that y_i wins against y_{i-1}. The result follows then by transitivity of the binary relation, as shown above . ■

We note here that the successive procedure need not implement the incomplete-information Condorcet winner even if agents are assumed to behave myopically in the sense described above.

4 Conclusion

We have studied static and dynamic voting procedures in a setting where agents have single-peaked, interdependent preferences over several alternatives, and we focused on the Condorcet winner. In contrast to the private values case, the complete-information Condorcet winner can be implemented via a static, direct revelation mechanism only if the set of signals available to the agents is restricted. In that case, dynamic procedures that invariably implement the complete-information Condorcet winner with private values differ in their information revelation and aggregation processes and yield different results - some positive, some negative - when preferences are interdependent. Finally, we have defined and show how to implement a new notion of incomplete-information Condorcet winner. This new notion coincides with the standard notion under complete information or under private values, but differs from it in our setting with interdependent values.

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