

Post-stratification = simple random sample + break up into strata after the fact

Some properties:

1) $[n_1, n_2, \dots, n_L]$ is now a random vector

$$\Pr(n_1 = m_1, \dots, n_L = m_L) = \frac{\binom{N_1}{m_1} \dots \binom{N_L}{m_L}}{\binom{N}{n}}$$

Pf: denominator = total number of possible samples overall
 numerator = number of possible samples satisfying the condition

2) Conditional on $\{n_1 = m_1, \dots, n_L = m_L\}$, we have stratified random sampling with sample sizes m_1, \dots, m_L . That is

- Simple random sample of size m_h in Stratum h .
- Strata independent of each other

Pf: We have $\Pr(S^0 = S) = \binom{N}{n}^{-1} \quad \forall S \in \mathcal{S}$.

Denote $\mathcal{S}(m_1, \dots, m_L)$ the set of samples with sizes m_1, \dots, m_L in the respective strata. We have, for any $S \in \mathcal{S}(m_1, \dots, m_L)$

$$\begin{aligned} & \Pr(S^0 = S \mid n_1 = m_1, \dots, n_L = m_L) \\ &= \Pr(\{S^0 = S\} \cap \{n_1 = m_1, \dots, n_L = m_L\}) / \Pr(n_1 = m_1, \dots, n_L = m_L) \\ &= \Pr(S^0 \in S) / \Pr(n_1 = m_1, \dots, n_L = m_L) \end{aligned}$$

[since $S^0 = S$ implies $\{n_1 = m_1, \dots, n_L = m_L\}$]

using Property ①

$$= \left[\frac{1}{\binom{N}{n}} \right] / \left[\binom{N_1}{m_1} \cdots \binom{N_L}{m_L} / \binom{N}{n} \right]$$

$$= \binom{N_1}{m_1}^{-1} \cdots \binom{N_L}{m_L}^{-1} \quad (*)$$

Now with simple random sampling of a sample S_h^0 of size h from Stratum h , we have

$$\Pr(S_h^0 = S_h) = \binom{N_h}{m_h}^{-1}$$

by standard random sampling theory.

The expression (*) is the product of these probabilities, which proves the claim.

3) If N_1, \dots, N_L are all large, then

$$\Pr(n_1 = m_1, \dots, n_L = m_L) \doteq \frac{n!}{m_1! \dots m_L!} W_1^{m_1} \dots W_L^{m_L}$$

Pf: We have

$$\Pr(n_1 = m_1, \dots, n_L = m_L)$$

$$= \binom{N_1}{m_1} \dots \binom{N_L}{m_L} / \binom{N}{n}$$

$$= \left[\frac{N_1!}{m_1! (N_1 - m_1)!} \dots \frac{N_L!}{m_L! (N_L - m_L)!} \right] / \left[\frac{N!}{n! (N - n)!} \right]$$

$$= \frac{n!}{m_1! \dots m_L!} \left[\frac{N_1!}{(N_1 - m_1)!} \dots \frac{N_L!}{(N_L - m_L)!} \right] / \left[\frac{N!}{(N - n)!} \right]$$

$$= \frac{n!}{m_1! \dots m_L!} \left\{ [N_1(N_1 - 1) \dots (N_1 - m_1 + 1)] \right.$$

$$[N_2(N_2 - 1) \dots (N_2 - m_2 + 1)]$$

$$\dots [N_L(N_L - 1) \dots (N_L - m_L + 1)] \left. \right\}$$

$$/ [N(N - 1) \dots (N - n + 1)]$$

$$= \frac{n!}{m_1! \dots m_L!}$$

$$\left\{ \left[\frac{N_1}{N} \frac{N_1-1}{N} \dots \frac{N_1-m_1+1}{N} \right] \right.$$

$$\left[\frac{N_2}{N} \frac{N_2-1}{N} \dots \frac{N_2-m_2+1}{N} \right]$$

$$\dots \left. \left[\frac{N_L}{N} \frac{N_L-1}{N} \dots \frac{N_L-m_L+1}{N} \right] \right\}$$

$$/ \left\{ \frac{N}{N} \frac{N-1}{N} \dots \frac{N-n+1}{N} \right\}$$

Now, if the N_h are all large, then

$$\frac{N_h - l}{N} \doteq \frac{N_h}{N} = W_h, \quad l = 1, \dots, m_L - 1,$$

and similarly $\frac{N-l}{N} \doteq 1, \quad l = 1, \dots, n.$

So we get

$$\Pr(n_1 = m_1, \dots, n_L = m_L)$$

$$\doteq \left(\frac{n!}{m_1! \dots m_L!} \right)$$

$$\left\{ [W_1 \dots W_1] [W_2 \dots W_2] \dots [W_L \dots W_L] \right\}$$

\uparrow \uparrow \uparrow
 m_1 times m_2 times m_L times

$$= \frac{n!}{m_1! \dots m_L!} W_1^{m_1} \dots W_L^{m_L}$$

$$4) \Pr(n_h = m_h) = \binom{N_h}{m_h} \binom{N-N_h}{n-m_h} / \binom{N}{n}$$

$$E[n_h] = n \frac{N_h}{N} = n W_h$$

$$\begin{aligned} \text{Var}(n_h) &= \frac{N_h(N-N_h)n(N-n)}{N^2(N-1)} \\ &= n \left(\frac{N-n}{N-1} \right) W_h (1-W_h) \end{aligned}$$

For $w_h = n_h/n$,

$$E[w_h] = W_h$$

$$\text{Var}(w_h) = \frac{1}{n} \frac{N-n}{N-1} W_h (1-W_h)$$

For N_h large

$$n_h \sim \text{Bin}(n, W_h)$$