בס״ד

<u>Fact</u>: If (S, d) is separable, then any open set G can be represented as a countable union of open balls:

$$G = \bigcup_{\ell=1}^{\infty} B(x_\ell, \delta_\ell) .$$

[<u>Pf</u>: Let S^* be a countable dense subset of S, and let $G^* = G \cap S^*$.

Let $x_1, x_2, \cdots$ be the elements of G^*.

Define

$$\delta_\ell = \sup \{ \delta > 0 : B(x_\ell, \delta) \subset G \} .$$

The fact that G is open implies that the set on the right hand side is nonempty.

Now it is easy to see that if $y \in B(x_\ell, \delta_\ell)$, then $y \in B(x_\ell, \delta)$ for some $\delta < \delta_\ell$. Thus

$$B(x_\ell; \delta_\ell) = \bigcup_{\delta < \delta_\ell} B(x_\ell; \delta)$$

The right hand side is a union of subsets of G, and hence is itself a subset of G.

Thus $B(x_\ell, \delta_\ell) \subset G$.]

Now let

$$A = \bigcup_{\ell=1}^{\infty} B(x_\ell, \delta_\ell).$$

The claim is that $A = G$.

It is clear that $A \subset G$.

To show $G \subset A$, we reason as follows. Let $x \in G$.
Then $\exists\, \varepsilon > 0$ such that $B(x, \varepsilon) \subset G$.
By denseness, $\exists\, x^* \in S^*$ such that $x^* \in B(x, \frac{\varepsilon}{2})$.
We have $x^* \in S^* \cap G \Rightarrow x^* = x_\ell$ for some ℓ.
Further:

- $d(x, x_\ell) < \frac{\varepsilon}{2} \Rightarrow x \in B(x_\ell, \frac{\varepsilon}{2})$

- If $y \in B(x_\ell, \frac{\varepsilon}{2})$, then $y \in B(x, \varepsilon) \subset G$ (using triangle inequality). Thus $B(x_\ell, \frac{\varepsilon}{2}) \subset G$, which implies $B(x_\ell, \frac{\varepsilon}{2}) \subset B(x_\ell, \delta_\ell)$ (by definition of δ_ℓ).

Thus $x \in A$.