Note on measurability in $C[a,b]$

Claim: Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X be a map from Ω to $C[a,b]$ (not necessarily measurable at this stage).

Suppose that, for any fixed u, $X(\omega, u)$ is a measurable map from Ω to $\mathbb{R}$.

Then X is measurable.

Pf: For given $x \in C[a,b]$ and $\delta > 0$, consider the closed ball $\overline{B}(x, \delta)$. We claim that $\{X \in \overline{B}(x,\delta)\} \in \mathcal{F}$. To see this, note that – since we are dealing with continuous functions – we can write

$$\{X \in \overline{B}(x,\delta)\} = \bigcap_{q} \{ |X(q) - x(q)| \leq \delta \}$$

where the intersection is over the rational numbers in $C[a,b]$.

Since $X(q)$ is measurable for each q, we have a countable union of measurable sets, which is measurable.

Now, an open ball can be written as a countable union of closed balls:

$$B(x,\delta) = \bigcup_{n=1}^{\infty} \overline{B}(x, \delta - \tfrac{1}{n})$$

Hence $\{X \in B(x, \delta)\}$ is measurable.

Next, since $C[a,b]$ is separable, any open set G can be written as a countable union of open balls. Hence

$$\{X \in G\} \in \mathcal{F}$$

for any open G. Since the open sets generate $\mathcal{S}$, it follows that

$$\{X \in A\} \in \mathcal{F}$$

for any $A \in \mathcal{S}$.