Transformations and Jacobians

Consider the following setup. We have a continuous random vector $\mathbf{X} = (X_1, X_2)$ with joint density function $f_{\mathbf{X}}(\mathbf{x})$. We have a function $\mathbf{G}(\mathbf{x}) = (G_1(\mathbf{x}), G_2(\mathbf{x}))$ mapping R^2 to itself. It is assumed that the function $\mathbf{G}(\mathbf{x})$ is invertible. That is, it is assumed that there exists a function $\mathbf{H}(\mathbf{y}) = (H_1(\mathbf{y}), H_2(\mathbf{y}))$ such that $\mathbf{H}(\mathbf{G}(\mathbf{x})) = \mathbf{x}$. We define the *Jacobian matrix* of the function $\mathbf{H}(\mathbf{y})$ to be the following matrix of partial derivatives:

$$\mathbf{J_H}(\mathbf{y}) = \begin{bmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} \\ \frac{\partial H_2}{\partial y_1} & \frac{\partial H_2}{\partial y_2} \end{bmatrix}.$$

Now define $\mathbf{Y} = \mathbf{G}(\mathbf{X})$, and let $f_{\mathbf{Y}}(\mathbf{y})$ denote its density function. We then have the following result.

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{H}(\mathbf{y}))\,|\det(\mathbf{J_H}(\mathbf{y}))|, \tag{1}$$

where "det" denotes determinant and the vertical bars are for absolute value. This result generalizes to the case of random vectors of arbitrary size (i.e., greater than two).

Example:

Suppose that X_1 and X_2 are independent random variables with $X_1 \sim \text{Exp}(\theta_1)$, $X_2 \sim \text{Exp}(\theta_2)$, and we want to find the joint density of the random variable $\mathbf{Y}$ defined by $Y_1 = X_1$, $Y_2 = X_2/X_1$. The joint density of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x_1, x_2) = \left[\theta_1 e^{-\theta_1 x_1}\right]\left[\theta_2 e^{-\theta_2 x_2}\right]. \tag{2}$$

The function $\mathbf{G}(\mathbf{x})$ is defined by

$$\begin{aligned} G_1(\mathbf{x}) &= x_1, \\ G_2(\mathbf{x}) &= x_2/x_1. \end{aligned}$$

The inverse function $\mathbf{H}(\mathbf{y})$ is given is obtained by solving the equations

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= x_2/x_1, \end{aligned}$$

to express (x_1, x_2) in terms of (y_1, y_2). We get

$$\begin{aligned} H_1(\mathbf{y}) &= y_1, \\ H_2(\mathbf{y}) &= y_1 y_2. \end{aligned}$$

The partial derivatives are as follows:

$$\begin{aligned} \frac{\partial H_1}{\partial y_1} &= 1, \\ \frac{\partial H_1}{\partial y_2} &= 0, \\ \frac{\partial H_2}{\partial y_1} &= y_2, \\ \frac{\partial H_2}{\partial y_2} &= y_1. \end{aligned}$$

So the Jacobian matrix is given by

$$\mathbf{J_H}(\mathbf{y}) = \begin{bmatrix} 1 & 0 \\ y_2 & y_1 \end{bmatrix}.$$

We have

$$|\det(\mathbf{J_H}(\mathbf{y}))| = |y_1| = y_1 \quad \text{(since in our case } y_1 > 0\text{)}.$$

Hence, (1) yields the following:

$$f_{\mathbf{Y}}(y_1, y_2) = f_{\mathbf{X}}(y_1, y_1 y_2) y_1.$$

Using the expression (2) for $f_{\mathbf{X}}(x_1, x_2)$ then gives

$$f_{\mathbf{Y}}(y_1, y_2) = y_1 \left[\theta_1 e^{-\theta_1 y_1}\right] \left[\theta_2 e^{-\theta_2 y_1 y_2}\right].$$