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ntral Limit Theorems in $D[0,1]$

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ummary. Let X be a stochastic process with sample paths in the usual
Skorohod space $D[0,1]$. For a sequence $\{X_n\}$ of independent copies of X, let
$S_n = X_1 + \cdots + X_n$. Conditions which are either necessary or sufficient for the
weak convergence of $n^{-1/2}(S_n - ES_n)$ to a Gaussian process with sample
aths in $D[0,1]$ are discussed. Stochastically continuous processes are con-
idered separately from those with fixed discontinuities. A bridge between
he two is made by a Decomposition central limit theorem.

ntroduction

aim of this paper is to study the central limiting behavior of right-
inuous stochastic processes with only jump discontinuities. For convenience
time set is assumed to be the unit interval. The sample paths of such
esses take their values in $D \equiv D[0,1]$, the space of functions which are right-
inuous on $[0,1)$ with left limits on $(0,1]$. Thus, interchangeably we can, and
peak of these processes as D-valued random variables. For details of D and
basic properties of the Skorohod topology on D, the reader is referred to
ngsley (1968) Chapter 3.

D-valued random variable X is said to *satisfy the CLT* if there exists a D-
ed Gaussian random variable Z which is the limit in distribution of the
ence of random variables

$$Z_n \equiv n^{-1/2}[X_1 + \cdots + X_n - E(X_1 + \cdots + X_n)],$$

e $X_1, X_2, \ldots$ are independent copies of X on the same probability space.
convergence will be denoted in symbols by $\mathscr{L}(Z_n) \to \mathscr{L}(Z)$ where $\mathscr{L}(Y)$
sents the law of Y. We will call Z the limiting Gaussian process of X.

he main problem is to find necessary and sufficient conditions for X to
y the CLT. Two necessary conditions are obvious:

$EX^2(t)$ is finite for each $t \in [0,1]$, as a result of the converse to the CLT in
se of real-valued random variables.

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(2) There exists a Gaussian process with sample paths in D and the covariance as X.

In Section 2 we show that for stochastically continuous processes the ab necessary conditions can be strengthened to requiring:

(1′) X is continuous in quadratic mean (CQM), i.e. in L^2.

(2′) There exists a sample-continuous Gaussian process with the covaria of X.

The latter condition is a consequence of

Theorem 1. *Any stochastically continuous Gaussian process with sample paths has sample paths a.s. in the space C of continuous functions.*

Since not every stochastically continuous process with finite second ments and sample paths in D is CQM, it is clear that not every D-val random variable satisfies the CLT. In fact, examples of uniformly bound continuous processes satisfying tight Lipschitz conditions but for which the C fails are discussed in Strassen and Dudley (1969), Dudley (1974), and Ha (1976).

In proving sufficient conditions it suffices to verify (2) and

(3) Tightness of $\{P_n \equiv \mathscr{L}(Z_n)\}$.

Our main result for stochastically continuous processes, which generali previous results of Phoenix and Taylor ((1973), pp. 207–208) and Taylor ((197 p. 2), is

Theorem 2. *Let X be a stochastic process with sample paths in D such that, for t in $[0,1]$, $EX(t)=0$ and $EX^2(t)<\infty$. Assume there exist nondecreasing co tinuous functions G and F on $[0,1]$ and numbers $\alpha>\frac{1}{2}$, $\beta>1$ such that for $0\leqq s\leqq t\leqq u\leqq 1$ the following two conditions hold:*

(i) $E(X(u)-X(t))^2 \leqq (G(u)-G(t))^\alpha$,

(ii) $E(X(u)-X(t))^2(X(t)-X(s))^2 \leqq (F(u)-F(s))^\beta$.

Then X satisfies the CLT in D and $\mathscr{L}(Z)(C)=1$.

An immediate corollary is the result of Fisz ((1959), p. 15) that the ri continuous version of a stochastically continuous process with independ increments and finite second moments satisfies the CLT.

As another consequence of Theorem 2 we obtain, in Section 3, the follow criterion for Markov processes to satisfy the CLT.

Theorem 3. *Let X be a Markov process with sample paths in D and nondecreasing continuous function. Suppose there exists $\beta>\frac{1}{2}$ such that for $s\leqq t$ either one of the following two conditions holds:*

(i) $\mathrm{ess}_\omega \sup E[(X(t)-X(s))^2 \mid X(s)] \leqq (F(t)-F(s))^\beta$;

(ii) $\mathrm{ess}_\omega \sup E[(X(t)-X(s))^2 \mid X(t)] \leqq (F(t)-F(s))^\beta$.

Then X satisfies the CLT in D.

These conditions are shown to be satisfied by a large class of stochastic continuous Markov processes with stationary transitions.

In the final section, processes with fixed discontinuities are discussed. Our pproach is two-fold. First a Decomposition CLT is proved which justifies, to me extent, separate treatments of stochastically continuous processes and ocesses with fixed discontinuities. For instance, processes with only a finite mber of fixed discontinuities can be immediately reduced to the stochastically ntinuous case.

Second, whenever a process X is continuous a.s. at all points except those in e at most countable set T of fixed discontinuities, it suffices to prove a CLT in (J_T), the space of continuous functions on a certain compact metric space J_T. tilization may now be made of known sufficient conditions for sample-ntinuous processes to satisfy the CLT such as the metric entropy condition in e Jain and Marcus theorem (1975) or the Lipschitz conditions on the second oments of increments appearing in Hahn (1977(a)), (1977(b)) and Hahn and lass (1977).

Application of the results in this paper necessarily requires verification of hether or not a process under consideration has sample paths in D. A haracterization of processes with sample paths in D in terms of their finite-imensional distributions can be found in Dubins and Hahn (1978). Separable ochastically continuous processes with independent increments (Gihman and korohod (1969) p. 168), separable stochastically continuous sub-martingales oob (1953), p. 361), and Markov processes under extremely broad conditions inney (1953)) all have versions with sample paths in D.

2. Stochastically Continuous Processes

stochastic process X is said to be *stochastically continuous at a point* t_0 in $,1]$ if for each $\varepsilon>0$, $P(|X(t)-X(t_0)|\geqq\varepsilon)\to 0$ as $|t-t_0|\to 0$. If X has sample ths in D, stochastic continuity at t_0 implies that $P(|X(t_0)-X(t_0-)|>0)=0$. Just tice that by stochastic continuity, $\lim_{s\uparrow t_0} P(|X(t_0)-X(s)|>k^{-2})\leqq k^{-2}$; hence, ere exists $s_k\uparrow t_0$ such that $P(|X(t_0)-X(s_k)|>k^{-2})\leqq 2k^{-2}$. Consequently, $\sum_1 P(|X(t_0)-X(s_k)|>k^{-2})<\infty$; so, by Borel-Cantelli, $P(|X(t_0)-X(s_k)|$ k^{-2} i.o.$)=0$. X is said to be *stochastically continuous* on $[0,1]$ if it is ochastically continuous at each point.

A stochastic process X is *continuous in quadratic mean* (CQM) if and only if r all t, $\lim_{s\to t} E(X(t)-X(s))^2=0$. It is well known that a Gaussian process is QM if and only if it is stochastically continuous.

roposition 1. *A stochastically continuous process X which satisfies the CLT in D CQM.*

oof. Let Z be the limiting Gaussian process which has sample paths in D. By e right-continuity of Z,

$$0=\lim_{s\downarrow t} E(Z(s)-Z(t))^2=\lim_{s\downarrow t} E(X(s)-X(t))^2;$$