# The Convolution Thm for the Normal Experiment

**Setup:** $W \sim N(h, \Sigma)$, $U \sim U(0,1)$ > indept

$A$ = fixed matrix

$$T = t(W, U)$$

**Def'n:** $T$ is called equivariant in law w.r.t. $Ah$ if the dist'n of

$$V = T - Ah$$

does not depend on $h$.

**Prop.:** If $T$ is equivariant in law w.r.t. $Ah$, then under $h = 0$ we have

$$T \overset{d}{=} T_1 + T_2,$$

where $T_1 \sim N(0, A \Sigma A^T)$ and $T_2$ is indept of $T_1$.

(I.e. $V \overset{d}{=} T_1 + T_2$)

בס"ד

<u>Proof:</u> Put $W^* = W - h \sim N(0, \Sigma)$.

We can then write

$$T = t(W^* + h, U)$$

Let $V^*$ be an r.v. with the dist'n of $T - Ah$ (which, by hypothesis, does not depend on $h$).

Then we have

$$t(W^* + h, U) - Ah \overset{d}{=} V^* \qquad \forall h .$$

Now, for $m = 1, 2, 3, \ldots$ let $H_m \sim U(-m, m)$ independent of $(W^*, U)$

We have

$$t(W^* + H_m, U) - AH_m \overset{d}{=} V^*$$

$$\Rightarrow \quad t(W^* + H_m, U) - A(W^* + H_m) + AW^* \overset{d}{=} V^*$$

i.e. $$\left[t(S_m, U) - AS_m\right] + AW^* \overset{d}{=} V^*,$$

where $S_m = W^* + H_m$.

We claim that $S_m$ and $W^*$ are asymptotically independent as $m \to \infty$. To see this, we reason as follows:

$$|Pr(S_m \in B, W^* \in C) - Pr(S_m \in B) Pr(W^* \in C)|$$

$$\leq |Pr(S_m \in B, W^* \in C) - Pr(H_m \in B) Pr(W^* \in C)|$$

$$+ |Pr(H_m \in B) - Pr(S_m \in B)| \, Pr(W^* \in C)$$

$$\leq 2 \int |Pr(w + H_m \in B) - Pr(H_m \in B)| \, f_{W^*}(w) \, dw$$

$$\leq 4 \, Pr(|W^*| > \sqrt{m})$$

$$+ 2 \sup_{|w| \leq \sqrt{m}} |Pr(w + H_m \in B) - Pr(H_m \in B)|$$

Take $B = (c, d]$. Then

$$Pr(w + H_m \in B)$$

$$= Pr(c - w < H_m \leq d - w)$$

$$= \frac{1}{2m} [\min(d - w, m) - \max(c - w, -m)]$$

As $m \to \infty$, with $w$ restricted to $[-\sqrt{m}, \sqrt{m}]$, we get

$$Pr(w + H_m \in B) = Pr(H_m \in B) = d - c.$$

Now let

$$\tilde{Z}_m = t(S_m, U) - A S_m$$

$$= t(W^* + H_m, U) - A(W^* + H_m)$$

Since $\tilde{Z}_m$ is a direct fn of $S_m$, we have that $\tilde{Z}_m$ and $W^*$ are asymptotically independent.

Let $Z_m$ be a r.v. that is exactly independent of $W^*$ and has the same dist'n as $\tilde{Z}_m$. Then

$$\sup_{A,B} \left| Pr(Z_m \in B \;\; W^* \in C) - Pr(\tilde{Z}_m \in B, W^* \in C) \right| \to 0 .$$

Define

$$\tilde{V}_m = \tilde{Z}_m + AW^* = t(W^* + H_m, U) - AH_m$$

$$V_m = Z_m + AW^*$$

We have

$$\sup_D \left| Pr(V_m \in D) - Pr(\tilde{V}_m \in D) \right| \to 0$$

and $\tilde{V}_m \overset{d}{=} V^*$, so $V_m \xrightarrow{d} V^*$.

Hence, for any $\varepsilon > 0$ we can find $K_\varepsilon$ such that

$$Pr(\| V_m \| \leq K_\varepsilon) \geq 1 - \varepsilon \quad \forall m.$$

In addition, we can find $K'_\varepsilon$ such that

$$\Pr(\|AW^*\| \le K'_\varepsilon) \ge 1-\varepsilon$$

Since $Z_m = V_m - AW^*$, we get

$$\Pr(\|Z_m\| \le K_\varepsilon + K'_\varepsilon) \ge 1-2\varepsilon \quad \forall m.$$

Thus, $Z_m = O_p(1)$.

So, by Prohorov's thm, there exists a subsequence $Z_{m_k}$ and an r.v. $Z$ such that

$$Z_{m_k} \xrightarrow{d} Z$$

For any $B$ with $\Pr(Z \in \partial B) = 0$ we have

$$\Pr(Z_m \in B,\ W^* \in C)$$

$$= \Pr(Z_m \in B)\Pr(W^* \in C)$$

$$\to \Pr(Z \in B)\Pr(W^* \in C)$$

Thus, $(Z_m, W^*) \xrightarrow{d} (Z, W^*)$,

where the r.v.'s $Z$ and $W^*$ are indept.

Recalling $V_m = Z_m + AW^*$ and $V_m \xrightarrow{d} V^*$, we get

$$V^* \overset{d}{=} T_1 + T_2,$$

where $T_1 = AW^* \sim N(0, A\Sigma A^T)$ and $T_2$ (a.k.a. $Z$) is independent of $T_1$.