Alternate Proof of Continuous Mapping Thm Using Portmanteau Thm

Suppose that $X_n \rightsquigarrow X$, and that h is a function such that $Pr(X \in D_h) = 0$.

For a given set $A \in \mathcal{S}'$ define $h^{-1}(A)$ as follows:

$$h^{-1}(A) = \{ x \in S : h(x) \in A \} .$$

(here $\mathcal{S}'$ denotes the Borel σ-algebra of (S', d')).

Now let F be a closed set in S'.

We claim that

$$\overline{h^{-1}(F)} \cap D_h^c \subset h^{-1}(F)$$

To see this, we reason as follows. Suppose

$$x \in \overline{h^{-1}(F)} \cap D_h^c$$

Then:

1) $x \in \overline{h^{-1}(F)} \Rightarrow$ There exists a sequence $\{x_n\}$ in $h^{-1}(F)$ such that $x_n \to x$ (by defn of closure)

2) $x \in D_h^c \Rightarrow h$ is continuous at x

3) ① and ② combined imply that $h(x_n) \to h(x)$.

4) The above plus the fact that F is closed implies that $h(x) \in F$. (Since $h(x_n) \in F$ $\forall n$ by defn)

So we get

$$\limsup \Pr(h(X_n) \in F)$$

$$= \limsup \Pr(X_n \in h^{-1}(F))$$

$$\leq \limsup \Pr(X_n \in \overline{h^{-1}(F)})$$

$$\leq \Pr(X \in \overline{h^{-1}(F)}) \qquad \text{(portmanteau)}$$

$$= \Pr(X \in \overline{h^{-1}(F)} \cap D_h^c) \qquad (\text{Since } \Pr(X \in D_h) = 0)$$

$$\leq \Pr(X \in h^{-1}(F))$$

$$= \Pr(h(X) \in F)$$

Hence, by the portmanteau thm, $h(X_n) \rightsquigarrow h(X)$.