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Stochastic Dominance and Prospect Dominance with Subjective Weighting Functions

HAIM LEVY
School of Business, The Hebrew University of Jerusalem, 91905 Israel

ZVI WIENER
School of Business, The Hebrew University of Jerusalem, 91905 Israel

Stochastic Dominance (SD) rules are used to divide the sets of all feasible uncertain prospects into efficient and inefficient sets (partial ordering). The SD rules (as well as the mean-variance rule) assume that investors agree on the available distributions of returns.

Laboratory experiments with and without real money repeatedly reveal that even if all subjects observe the same pair of cumulative distributions F and G, they act as if they were other cumulative probability functions F^* and G^* different for different investors. Namely, the subjects assign (subjective) weights to the various probabilities. In their breakthrough article Kahneman and Tversky [1979] suggest that in making decisions under uncertainty, the subjects apply a monotonic transformation $\pi(p)$ where p are the probabilities, and investors make decisions by comparing $\pi(p)$ corresponding to the two distributions under consideration rather than by comparing the true probabilities, p, themselves. In a later paper, Tversky and Kahneman [1992] suggest that a transformation is done on the cumulative probability F rather than on probability p, namely, investors compare $F^* = T(F)$ and $G^* = T(G)$ where T is a monotonic non-decreasing transformation. T can be interpreted either as a misjudgment or as a subjective revision of probabilities which varies across investors. In such cases, even though all subjects observe the same pair of distributions F and G, the distributions F^* and G^* may vary across investors. Hence stochastic dominance may lose ground because each investor has his/her own subjective feasible set, hence, in general each investor has his/her efficient set. In fact this is also true of the mean-variance efficiency analysis introduced by Markowitz [1952]. The reason for the disagreement between investors on F^* and G^* is that each subject may have his subjective transformation $T_i(F)$ and $T_i(G)$ where the subscript i corresponds to the i-th investor. Thus even if the objective feasible set is identical for all investors, the subjective feasible set (which is the relevant one for decision making) may vary among investors.

While our paper is inspired by the Kahneman and Tversky's (K&T) weighting function, our analysis of the impact of subjective transformation on SD relationship is quite general and includes other cases of probabilities changes as well. For example, suppose that all investors observe a set of data forming F and G and are in complete agreement regarding these two distributions. Then a new piece of a non firm specific information arrives, for example that the Federal Reserve Board increased the interest rate, or a new government is elected, etc. Each investor may have a different interpretation of the new information.

Note that the new information is not firm specific, hence it is reasonable to assume that the transformation T_i is the same for the *i*-th investor across all assets. However, if the information is firm specific, e.g. an increase of dividends or earnings announcement, then T_i can not be the same for all assets. Yet recall that even when there is no such a new information simply following K&T, each individual adjusts the probability subjectively in some particular manner, using his/her transformation T_i. The investors make their choices based on $T_i(F)$ and $T_i(G)$, hence $T_i(F)$ and $T_i(G)$ are the relevant distributions for the *i*-th investor's decision, where T_i is the *i*-th investor's transformation of F and G due to the new information. Then, it is questionable whether the dominance results with F and G are intact for F_i^* and G_i^*.

The purpose of this paper is to analyze under what set of transformations $\mathbf{T_1}$ the First Degree Stochastic Dominance (FSD) is unaffected (we use bold letters for sets to distinguish between a set of transformations and a particular one). Similarly, we check the set of transformations $\mathbf{T_2}$ which do not affect Second Degree Stochastic Dominance (SSD). Note that if the SD (partial ordering) is not affected by a set of transformations, then all investors can employ the same transformation or various transformations as long as they belong to the set we consider.

The structure of the paper is as follows: In section 1 we present the well-known stochastic dominance rules as well as the new Prospect Stochastic Dominance (PSD) rule. In section 2 we show that for each type of stochastic dominance there exists a corresponding class of transformations which do not change the dominance relation. Concluding remarks are given in section 3.

There is a closely related approach developed in Zilcha [1987], Chew, Karni and Safra [1987], Safra and Zilcha [1988], Chew and Zilcha [1990], which is based on comparison of the efficient set when we consider investors which follow the stochastic dominance rules but have state dependent utility functions. Our settings represent a particular case of this general approach which enables us to obtain more precise results while focusing on the Prospect Theory approach to decision making. However, some results are common in the two approaches. For example, Chew, Karni and Safra show that a concave utility function combined with an increasing, concave transformation function is equivalent to risk aversion.

1. Stochastic dominance rules and the prospect stochastic dominance rule

In this section we briefly review the various decision rules analyzed in this paper. Two rules discussed in this paper (FSD, SSD) have existed in the literature for years (see Levy [1992]), but the SSD* and PSD rules are new and will be described below.

Let us first define several classes of preferences (sets of utility functions) corresponding to the various stochastic dominance rules. In each case we assume the utility function to be continuous and piecewise smooth. To distinguish between a set and its elements we use bold letters for sets.

- $\mathbf{U_1}$ is the set of increasing utility functions (rational investors). $U \in \mathbf{U_1}$ is equivalent to $U'(x) \geq 0$ for all[1] x.
- $\mathbf{U_2}$ is the set of increasing and concave utility functions (risk averters). $U \in \mathbf{U_2}$ is equivalent to $U'(x) \geq 0$ and $U''(x) \leq 0$ for all x.
- $\mathbf{U_2^*}$ is the set of increasing and convex utility functions (risk seekers). This corresponds to all investors that like risk and hence, prefer an uncertain income over its certain mean. $U \in \mathbf{U_2^*}$ is equivalent to $U'(x) \geq 0$ and $U''(x) \geq 0$ for all x.
- $\mathbf{U_p}$ is the set of all S-shaped utility functions as suggested by K&T [1979]. $U \in \mathbf{U_p}$ is equivalent to $U'(x) \geq 0$ for all x, and $U''(x) \leq 0$ for all $x > 0$, and $U''(x) \geq 0$ for all $x < 0$. This means risk seeking for losses and risk aversion for gains.

We first study the SD rules in terms of cumulative distributions and then in terms of the inverse functions known as the quantiles of the distribution. We need both forms in the subsequent analysis.

1.1. SD: Cumulative distributions

Theorem 1. Let F and G be two cumulative distributions of two uncertain prospects, x the uncertain return and U a utility function. Then F dominates G for all rational investors ($U' \geq 0$) iff,

$$F(x) \leq G(x) \text{ for all } x. \tag{1}$$

This rule is called First Degree Stochastic Dominance (FSD) (for discussion see Levy [1992]). Note that we ignore the initial wealth w because it induces a constant shift in F and G without affecting the stochastic dominance (SD) relationship. The same holds for SSD.

Theorem 2. Let F, G and x be defined as above. Then F dominates G for all risk averse investors ($U' \geq 0$, $U'' \leq 0$) iff

$$\int_{-\infty}^{x} (G(t) - F(t))dt \geq 0 \text{ holds for all } x. \tag{2}$$

This rule is called Second Degree Stochastic Dominance (SSD) (the discussion of SSD can be found in Levy [1992]).

Theorem 3. Let F, G and x be defined as above. Then F dominates G for all risk-seekers ($U' \geq 0$, $U'' \geq 0$) iff

$$\int_x^\infty (G(t) - F(t))dt \geq 0 \text{ for all } x. \tag{3}$$

We call this rule also a second degree stochastic dominance rule (because a restriction of second degree is imposed on U) but we denote it SSD* to distinguish from SSD. This rule is valid for risk lovers. To prove the SSD* rule we note that $U(x)$ is convex increasing if and only if $-U(-x)$ is concave increasing. Then (x SSD* y) is equivalent to ($-y$ SSD $-x$) and the integral condition (2) reduces immediately to the integral condition (3)[2]. While SSD* by itself is not very important it is used in the proof of the new investment rule which we call Prospect Stochastic Dominance (PSD), to which we turn next.

Theorem 4. Let F, G and x be as above and U be a utility function. Then F dominates G for all $U \in \mathbf{U_p}$, (the class of all S-shaped utility functions) iff[3]

$$\int_y^x (G(t) - F(t))dt \geq 0 \text{ for all pairs } x > 0 \text{ and } y < 0. \tag{4}$$

Note that for all $U \in \mathbf{U_p}$, we have a risk seeking as well as a risk aversion segment. Indeed, to prove PSD we need to use both SSD and SSD*. This result follows immediately by considering separately the gains (integral over $[0, x]$) and the losses (integral over $[y, 0]$).

1.2 Stochastic dominance: The quantile approach

In the following sections we analyze the impact of various transformations on the Stochastic Dominance rules. Some of these results can be derived more easily if we work not with the cumulative distributions but rather with their quantiles. The quantile approach for dominance analysis was introduced first in Levy [1973] for lognormal distributions and further developed by Kroll and Levy [1978] for unrestricted distributions. We briefly review the SD rules in the quantile approach.

The p-th quantile denoted by $Q_F(p)$ is defined by the following equation $P_F(x \leq Q_F(p)) = p$, and $Q_G(p)$ is defined similarly. Since the F function is a weakly increasing function, its inverse Q_F is defined as a multi-valued function. In our proofs we assume that all cumulative probability functions are strictly increasing (which guarantees that the inverse function Q_F is a well defined single-valued function). This assumption simplifies the proofs; however the results are valid even without it[4]. Therefore in our settings the following identities hold $p = F(Q_F(p)) = F(x)$ and $x = Q_F(F(x))$. The function Q_F is the inverse of F, it maps the interval $[0, 1]$ into the real axis. Figures 1a and 1b illustrate a cumulative distribution $F(x)$ and its quantile (the inverse function) $Q_F(p)$.

The following rules give SD criteria in terms of quantiles.

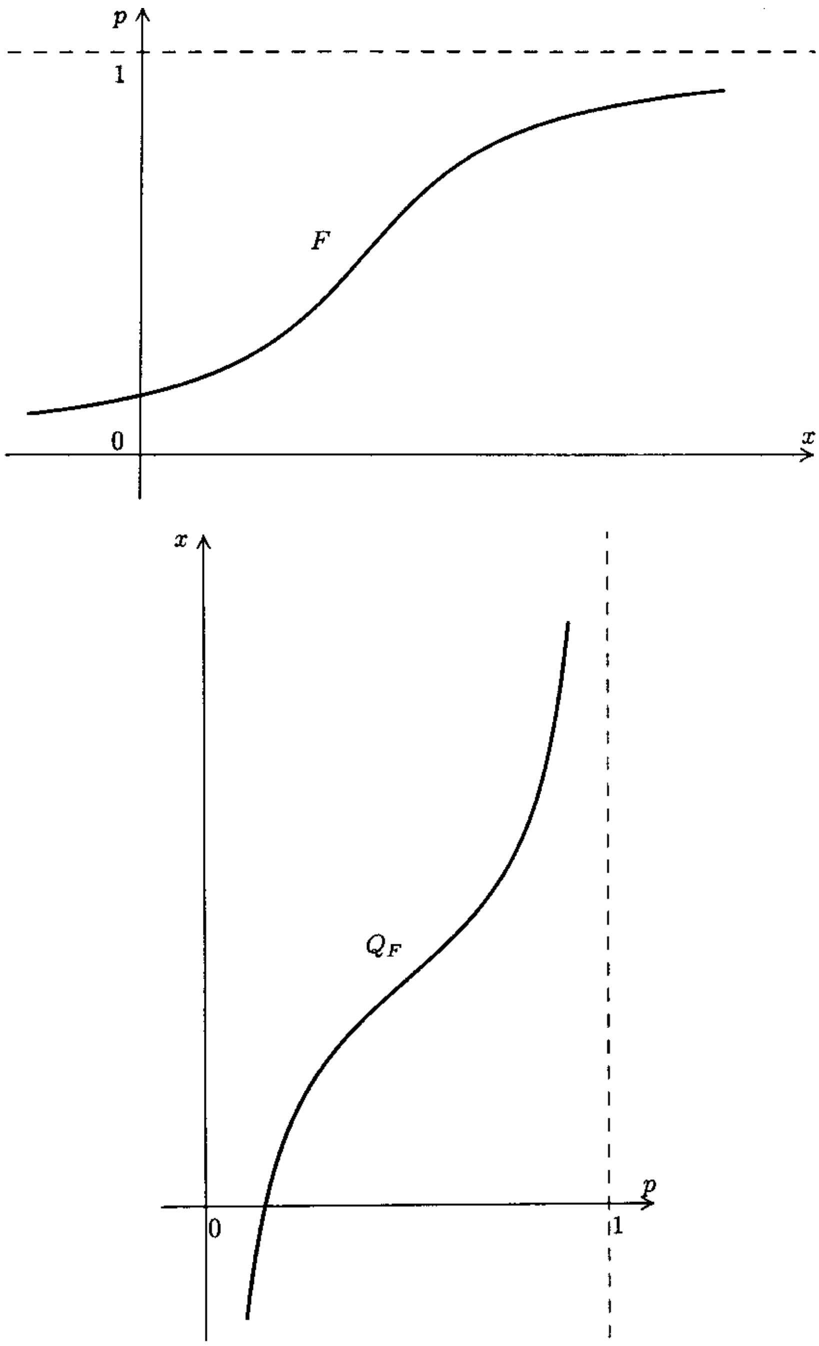

Figure 1 a. Standard approach. Cumulative distribution $p = F(x)$. b. Quantile approach $x = Q_F(p)$.

Quantile SD rules:

a. F dominates G by FSD is equivalent to

$$Q_F(p) \geq Q_G(p) \text{ for all } 0 \leq p \leq 1. \tag{5}$$

b. F dominates G by SSD is equivalent to

$$\int_0^p (Q_F(v) - Q_G(v))dv \geq 0 \text{ for all } 0 \leq p \leq 1. \tag{6}$$

c. F dominates G by SSD* is equivalent to

$$\int_p^1 (Q_F(v) - Q_G(v))dv \geq 0 \text{ for all } 0 \leq p \leq 1. \tag{7}$$

d. F dominates G by PSD is equivalent to

$$\int_{p_1}^{p_2} (Q_F(v) - Q_G(v))dv \geq 0 \text{ for all } 0 \leq p_1 \leq p_{F0} \leq p_{G0} \leq p_2 \leq 1. \tag{8}$$

where $F(0) = p_{F0}$ and $G(0) = p_{G0}$, as shown in Figure 2.

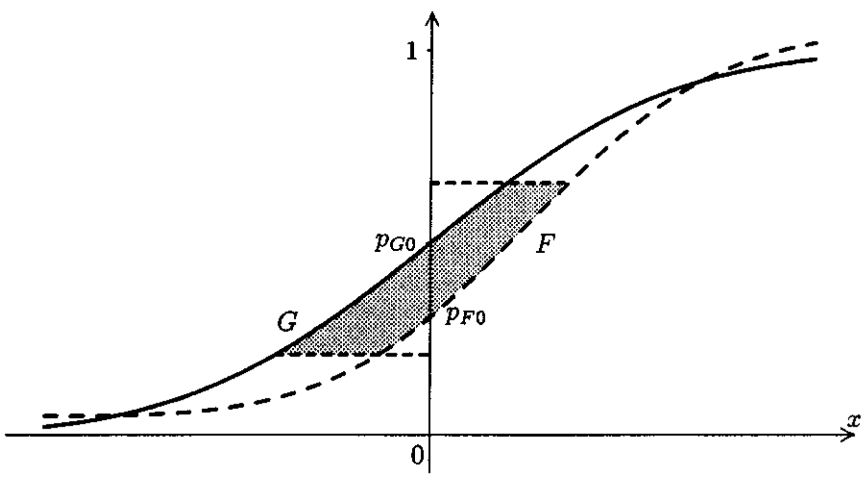

Figure 2. The qantile rule for PSD.

The proof of *a* and *b* can be found in Kroll and Levy [1978]. The proof of *c* and *d* can be constructed along the same lines as in *a* and *b* and for the sake of brevity we do not give it here. However we outline the proof later in the paper when we analyze the effect of various transformations on SD rules.

2. Weighting functions: Mental transformation of probabilities

2.1. Definitions and examples

In the 1979 paper K&T suggest a transformation of probabilities $\pi(p)$. However they show that such a transformation may violate First Degree Stochastic Dominance (FSD), hence in a later paper (see T&K [1992]) they suggest a transformation of cumulative probability functions which guarantees that if FSD applies before the transformation, this dominance is preserved by the transformation.

K&T claim that the *i*-th subject (investor) observes F but makes choices by $F_i^* = T_i(F)$. We deal with such transformations but our analysis is not limited to irrational probability calculations that investors make. We extend the analyses also to rational transformations when F_i^* is the subjective probability distribution derived from F when new non firm specific information is released. We have the subscript i because each investor may have his interpretation of the news.

We may have a situation were F dominates G (by some degree), yet after the transformation $T_i(F)$ may not dominate $T_i(G)$. We demonstrate below two alternative transformations corresponding to two individuals. Later on in the paper we will show that both transformations keep FSD but only one of them keeps SSD (the one that is concave). To be more specific, suppose that F dominates G by SSD. Then we show that for one investor $T_i(F)$ dominates $T_i(G)$ by SSD, but for another investor (with transformation T_j) this does not generally hold. We demonstrate below two transformations T_i and T_j such that application of them does not change FSD but SSD is not generally preserved.

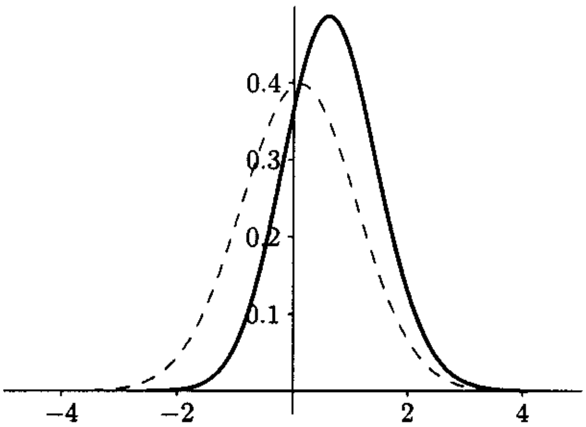

Figure 3a. Example of a transformation $T(p) = p^2$.

To illustrate how the transformations work we use the two transformations $T(F) = F^2$ and $T(F) = F^{1/2}$. Apply them for example to a normal distribution. Figure 3a demonstrates what happens with the normally distributed variable after the first transformation (the transformed distribution is not normal any more, but the area below the line is still equal to unity).

The dashed line represents the normal density function before transformation $\varphi(x)$ and the solid line represents the density after the transformation which was calculated as

$$\frac{\partial}{\partial x} \mathrm{T}\left(\int_{-\infty}^{x} \varphi(y)dy\right) = \frac{\partial}{\partial x}\left(\int_{-\infty}^{x} \varphi(y)dy\right)^2$$

This formula gives the density (derivative with respect to x) of the transformation (T) of the cumulative (integral) distribution. This transformation increases the mean from 0 to $1/\sqrt{2\pi}$, and reduces the variance from 1 to $1 - 1/\pi$ in this case[5]. Since $F^2 \leq F$ for $0 \leq F \leq 1$, $T(F)$ dominates F by FSD. Note that this transformation is convex $T'' > 0$ and its properties will be described in Theorem 7 below. Thus, such a transformation "subjectively" improved the investor's welfare, i.e. he/she believes that investment is better than what it actually is.

Let us turn now to another transformation that "subjectively" decreases the expected utility. Take the transformation $T(F) = F^{1/2}$. When applied for example to the standard normal density we obtain F and $T(F)$ as shown in the following figure:

The solid line represents the density after the transformation. The cumulative distribution for $T(F) = F^{0.5}$ is given by:

$$\frac{\partial}{\partial x} T\left(\int_{-\infty}^{x} \varphi(y)dy\right) = \frac{\partial}{\partial x}\left(\int_{-\infty}^{x} \varphi(y)dy\right)^{1/2}.$$

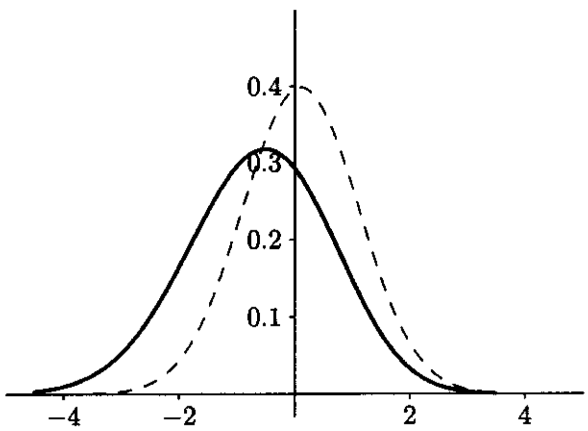

Figure 3b. Example of a transformation $T(p) = p^{1/2}$.

Here the mean after the transformation is −0.704307 and the variance is 1.55723 (before the transformation the mean was 0 and variance 1). It is easy to show that F dominates $F^* = T(F)$ by FSD. Thus the investor thinks that the distribution is not as good as it actually is. An interesting question is what happens with the dominance relation under this transformation, in other words if F dominates G, what can we say about $T(F)$ and $T(G)$? Theorem 6 below gives the complete answer to this question.

We analyze the impact of a transformation T on FSD, SSD, SSD* and PSD relationships. We do not distinguish whether the source of the transformation is mental (and hence irrational) or a rational adjustment to a new information item. We do not deal with a specific transformation, but with classes of transformations that are similar in concept to the classes of utility functions defined above. We introduce the following classes of transformations[6]:

- $\mathbf{T_1}$ the class of monotonic (nondecreasing) transformations[7], i.e. $T \in \mathbf{T_1}$ means $T'(p) \geq 0$ for all p.
- $\mathbf{T_2}$ the class of monotonic and concave transformations, i.e. $T \in \mathbf{T_2}$ means $T'(p) \geq 0$ and $T''(p) \leq 0$ for all p.
- $\mathbf{T_2^*}$ the class of monotonic and convex transformations, i.e. $T \in \mathbf{T_2^*}$ means $T'(p) \geq 0$ and $T''(p) \geq 0$ for all p.
- $\mathbf{T_p}$ the class of monotonic transformations that are concave for gains and convex for losses.

We will prove that FSD is preserved under $\mathbf{T_1}$, SSD is preserved under $\mathbf{T_2}$, SSD* is preserved under $\mathbf{T_2^*}$, and PSD is preserved under $\mathbf{T_p}$.

2.2. The effect of transformations on dominance relationship

FSD with transformations $\mathbf{T_1}$

Theorem 5. Any monotonic transformation $T \in \mathbf{T_1}$ preserves the first degree stochastic dominance. In other words if prospect F dominates G by FSD before transformation T, the dominance is intact after the transformation as well.

As $T(\bullet)$ is monotonic, the proof is obvious. Note that the inverse of theorem 5 is also true: if $F^* = T(F)$ dominates $G^* = T(G)$ for every $T \in \mathbf{T_1}$ then F dominates G. This may be seen by using the identical transformation (which belongs to the class $\mathbf{T_1}$) which does not change the distribution and therefore $F^* = F$ and $G^* = G$. Since the distribution is unchanged, here too F dominates G.

The two transformations $F^* = F^2$ and $F^* = F^{1/2}$ described in section 2.1 belong to $\mathbf{T_1}$, hence the FSD is preserved by both transformations for all distributions.

SSD with transformations $\mathbf{T_2}$ Now we turn to Second Degree Stochastic Dominance (SSD) which defines preferences of investors with a concave utility function (i.e. risk averters).

Theorem 6. A monotonic transformation preserves SSD iff it is concave (i.e. $T \in \mathbf{T_2}$).

Proof. In this proof we employ the quantile approach. Assume that the transformation T is smooth (at least twice differentiable)[8] and that F dominates G by SSD. Then $D(x) = \int_{-\infty}^{x} (G(t) - F(t))dt \geq 0$ for all x. This implies (see quantile SD rule, eq. 6) that $\int_{0}^{p} (Q_F(v) - Q_G(v))dv \geq 0$ for all $0 \leq p \leq 1$. We need to prove that $\int_{-\infty}^{x} (T(G(t)) - T(F(t)))dt \geq 0$ for all x, or that $\int_{0}^{p} (Q_{T(F)}(v) - Q_{T(G)}(v))dv \geq 0$ for all $0 \leq p \leq 1$.

Let us first explain graphically the main idea of the proof and then proceed with the formal proof. In Figure 4 we have the original distributions F and G and the transformed distributions $T(F)$ and $T(G)$. For any horizontal line the cumulative probability is the same and the values on the horizontal axis give the corresponding quantiles (see Figure 4). For example x_1 is the p_1-th quantile of distribution G and the p_2 quantile of distribution $G^* = T(G)$. Similarly x_2 is the p_1 quantile of distribution F and p_2 quantile of $F^* = T(F)$. Because for a given value $p_1 = F(x_1) = G(x_2)$ the transformations $T(F)$ and $T(G)$ have the same values and therefore the line CD must be parallel to the line AB and hence the distance between A and B is equal to the distance between C and D. Thus we have for $p_2 = T(p_1)$ that $|AB| = |CD|$ or

$$Q_F(p_1) - Q_G(p_1) = Q_{T(F)}(p_2) - Q_{T(G)}(p_2).$$

Thus we can use the transformed differences, such as $|CD|$ instead of $|AB|$. Recall that for any value such as p_1 there is a corresponding value like p_2 such that $|AB| = |CD|$. Denote by $H(p)$ the difference in the quantiles of $F^* = T(F)$ and $G^* = T(G)$, and by $h(p)$ the difference between the quantiles of the original distributions F and G. Then by the assumptions of the theorem it is given that $\int_{0}^{p} h(v)dv \geq 0$ for every p and we need to prove that $\int_{0}^{p} H(v)dv \geq 0$ for every p. However because for every value p_1 there is the corresponding value $p_2 = T(p_1)$ such that $h(p_1) = H(p_2)$ we can switch from $H(p_2)$ to $h(p_1)$ and to express the inequality in terms of the original distributions F and G (with an appro-

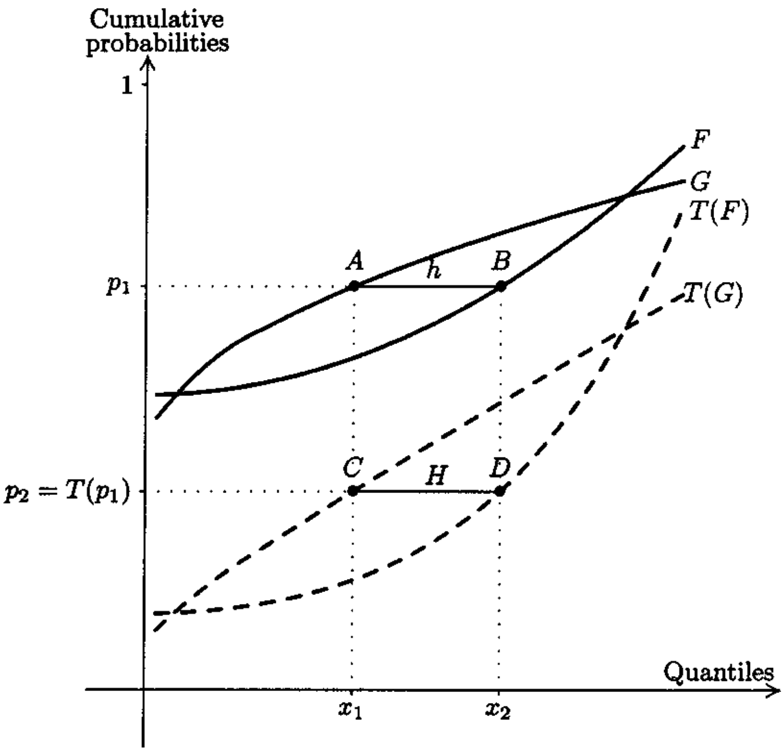

Figure 4. The distributions F and G before transformation T and after.

priate change of limits). To preserve the dominance we need $T'' \leq 0$ as shown below. This is because we actually have to sum up the areas between the curves F and G and not the intervals like $|AB|$. However the areas are stretched by T''.

Let us turn now to the formal proof. We define

$$h(p) \equiv Q_F(p) - Q_G(p)$$

$$H(p) \equiv Q_{F^*}(p) - Q_{G^*}(p)$$

For the difference in the quantiles of distributions F^* and G^* we have

$$H(p) = Q_F(T^{-1}(p)) - Q_G(T^{-1}(p)) = h(T^{-1}(p))$$
.

Therefore we have the following identity $\int_0^p H(v)dv = \int_0^p h(T^{-1}v)dv$. Let us introduce a new variable: $w = T^{-1}v$; this implies $v = Tw$, $dv = T'(w)dw$. Then we get

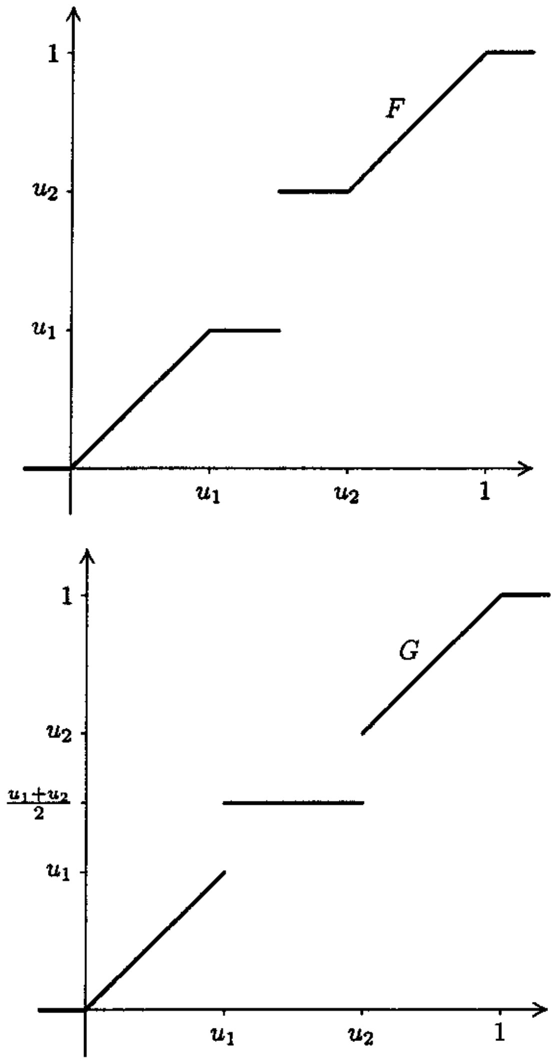

Figure 5 a. Cumulative distribution *F* before transformation. b. Cumulative distribution *G* before transformation.

$$\int_0^p h(T^{-1}v)dv = \int_0^{T^{-1}p} h(w)T'(w)dw$$

Integrating by parts the term on the right hand side we obtain

$$T'(w)\int_0^w h(u)\,du\Big|_0^{T^{-1}p} - \int_0^{T^{-1}p}\left(\int_0^w h(u)du\right)T''(w)dw =$$

$$T'(T^{-1}p)\int_0^{T^{-1}p} h(u)du - \int_0^{T^{-1}p}\left(\int_0^w h(u)du)\right)T''(w)dw \geq 0$$

since $T' \geq 0$, $T'' \leq 0$ (because we consider $T \in \mathbf{T_2}$) and the integral of h from 0 to any number between 0 and 1 is non-negative (by the assumption of the theorem that F dominates G). This proves that $\int_0^p H(v)dv \geq 0$ for all values of p which implies (quantile SD rule, eq. 6) that $\int_{-\infty}^x (T(G(t)) - T(F(t)))dt \geq 0$ for all values of x. Showing that there is at least one value for which the inequality is strict is trivial hence for the sake of brevity is omitted. This proves that if F dominates G by SSD then also $F^* = T(F)$ dominates $G^* = T(G)$ as long as $T \in \mathbf{T_2}$.

We now prove conversely that any monotonic transformation of the cumulative probability preserving Second Degree Stochastic Dominance must be concave. Assume that there is a transformation T which is monotonic, but is not concave and assume that it preserves the SSD rule. Then there is a region in which $T''(u) > 0$ for all $p \in [p_1, p_2]$ (recall that $0 \leq p_1 < p_2 \leq 1$). Consider the following prospects F and G. They coincide for all values below p_1 and above p_2 however are different between these two values. Set

$$F(t) = \begin{cases} 0, \textit{ if } t < 0, \\ t, \textit{ if } 0 \leq t < p_1, \\ p_1, \textit{ if } p_1 \leq t < 0.5(p_1 + p_2), \\ p_2, \textit{ if } 0.5(p_1 + p_2) \leq t < p_2, \\ t, \textit{ if } p_2 \leq t < 1, \\ 1, \textit{ if } 1 \leq t. \end{cases} \qquad G(t) = \begin{cases} 0, \textit{ if } t < 0, \\ t, \textit{ if } 0 \leq t < p_1, \\ 0.5(p_1 + p_2), \textit{ if } p_1 \leq t < p_2, \\ t, \textit{ if } p_2 \leq t < 1, \\ 1, \textit{ if } 1 \leq t. \end{cases}$$

Then after the transformation the cumulative distributions $F^* = T(F)$ and $G^* = T(G)$ still coincide outside the region $[p_1, p_2]$. Inside this region they have the form:

$$F^*(t) = \begin{cases} T(p_1), \textit{ if } p_1 \leq t < 0.5(p_1 + p_2), \\ T(p_2), \textit{ if } 0.5(p_1 + p_2) \leq t < p_2, \end{cases} \qquad G^*(t) = T(0.5(p_1 + p_2)), \textit{ if } p_1 \leq t < p_2,$$

Convexity of T in this region implies $T(0.5(p_1 + p_2)) < (T(p_1) + T(p_2))/2$ and after the transformation this region looks like as shown in Figure 6.

This implies that there is no SSD any more, contradicting our assumption about T. Once again let us refer to the two examples given in section 2.1. The transformation F^*

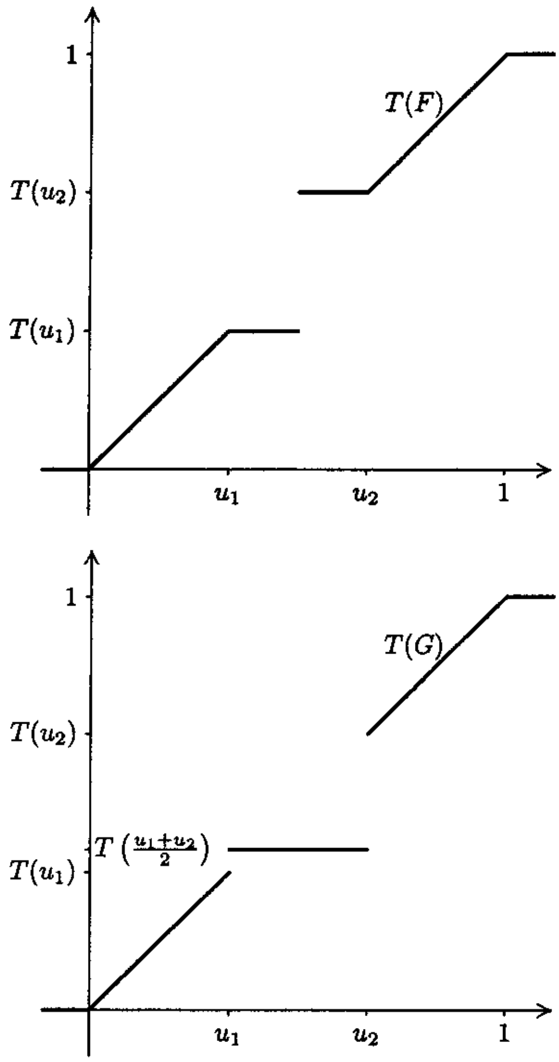

Figure 6 a. The cumulative distribution *T*(*F*) after the transformation. b. The cumulative distribution T(G) after the transformation.

$= F^{1/2}$ is concave, it belongs to $\mathbf{T_2}$ and hence preserves the SSD rule. However the transformation $F^* = F^2$ is convex, it does not belong to $\mathbf{T_2}$ and hence may not preserve the SSD.

SSD with transformations* $\mathbf{T}_2^*$

Theorem 7. A monotonic transformation preserves SSD* iff it is convex (i.e. $T \in \mathbf{T}_2^*$).

This theorem can be proved similar to the Theorem 3 by introducing an artificial risk averse investor and replacing gains by losses. Alternatively one can repeat the quantile approach that we have discussed in the Theorem 6. Note that in the example given in section 2.1, $F^* = F^2$ belongs to $\mathbf{T}_2^*$ and preserves the SSD* rule, though $F^* = F^{1/2}$ does not.

PSD with transformations $\mathbf{T_p}$ Many normative economic models assert how investor should behave. Positive economic models[10] assert how investors do behave. These positive models reveal strong evidence that,

a. The utility function of investors is *S*-shaped. By *S* shaped we mean increasing and concave for gains, increasing and convex for losses, the inflection point is at the current wealth.
b. Investors apply a weighting function to *F*, namely they make choices by comparing $F_i^* = T_i(F)$ with $G_i^* = T_i(G)$ and not by comparing *F* with *G*.

In this section we analyze the $\mathbf{U_p}$ class of utility functions which contains all possible *S*-shaped utility functions. We show that under the class of transformations $\mathbf{T_p}$, the PSD relationship remains. Since an *S*-shaped utility function causes risk seeking in the negative domain and risk aversion in the positive, this section relies on the results found in the previous sections.

Theorem 8. *F* dominates *G* by PSD iff every transformation $T \in \mathbf{T_p}$ preserves this dominance.

Proof. Assume that *F* dominates *G* by PSD, then by theorem 4 this implies $\int_y^x (G(t) - F(t))dt \geq 0$ for all $y < 0 < x$. Since the set $\mathbf{T_p}$ consists of transformations that are monotonic and concave (convex) in the domain of gains (losses) we conclude that each integral $\int_y^0 (G(t) - F(t))dt \geq 0$ and $\int_0^x (G(t) - F(t))dt \geq 0$ preserves its sign under the corresponding transformation hence: $\int_y^0 (G^*(t))\, dt \geq 0$ and $\int_0^x (G^*(t) - F^*(t))dt \geq 0.$ This implies that we have $\int_y^x (G^*(t) - F^*(t))dt \geq 0$, then using the theorem 4 we get the PSD rule again. To prove necessity we use the fact that among the class of $\mathbf{T_p}$ transformations there is a transformation that almost does not change the distribution. This completes the proof.

Note that the two transformations given in the section 2.1 do not belong to $\mathbf{T_p}$, hence do not preserve the PSD.

3. Concluding remarks

Efficiency analysis of investments and in particular stochastic dominance partial ordering assumes homogeneity of expectations regarding the probability distribution. Experimental studies reveal that even if all investors observe the same distribution of returns, they perform a mental transformation of the cumulative distribution. Each investor has his/her specific transformation and as a result Stochastic Dominance (SD) rules may lose ground. In addition it is found that the utility function is *S*-shaped (it is also called an *S*-shaped value function).

We develop in this paper a dominance rule called Prospect Stochastic Dominance (PSD) corresponding to the *S*-shaped value functions. We find a wide range of probability transformations under which the first degree, the second degree and the prospect stochastic dominance are intact.

It is shown in this paper that SD rules are intact as long as some constraints are imposed on the transformations $T(\bullet)$ of the cumulative probabilities: monotonicity of the transformation is required for FSD to be intact, monotonicity and concavity of $T(\bullet)$ to keep SSD intact, and convexity of $T(\bullet)$ for losses and concavity of $T(\bullet)$ for gains to keep PSD intact.

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Notes

1. When we say that a function $f(x) \geq 0$ (or $f(x) \leq 0$) for all x, we mean that this function is not negative (not positive) for all x and a strong inequality holds at least for one value of x. This important convention is used for the rest of the paper for all utility functions and the various decision rules before and after transformations.
2. We wish to thank the anonymous referee for pointing out this elegant proof. However this approach should be used carefully in the case of discrete events. An alternative proof (longer, but valid also in the discrete case) is available from the authors by request.
3. We do not ignore another requirement of prospect theory, which is that the slope of the value function is bigger for losses than for gains. Our proof is valid for all *S*-shaped utility functions including the functions with this specific property.
4. This is a technical condition. It can be easily resolved either by choosing an appropriate value of the (multi-valued) inverse functions Q_F or by a tiny change of all probabilities which does not influence preferences but assigns a positive probability for every interval of positive length.
5. Calculated with the *Mathematica* program.
6. Recall that by writing $f(x) \geq 0$ ($f(x) \leq 0$) for all x we actually mean that it is not negative (not positive) for all x and positive (negative) at least for one value of x.
7. Throughout the article we assume $T(0) = 0$ and $T(1) = 1$, so the monotonic transformation is nondecreasing.
8. If not one can always approximate T by a smooth transformation and repeat the same arguments.

9. We wish to emphasize that although $H(p_2) = h(p_1)$ for all pairs $p_2 = T(p_1)$ the integration range changes and we need to prove that dominance of F over G implies that F^* dominates G^* as well. To see this note that the area between F and G is not equal to the area between F^* and G^* (see Figure 4), however for every value p_1 there exists the corresponding $p_2 = T(p_1)$ such that $h(p_1) = H(p_2)$.
10. Like the one advocated by Friedman & Savage [1948] and Kahneman & Tversky [1979], [1992] and Tversky and Wakker [1995].

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