JOURNAL OF DIFFERENTIAL EQUATIONS **103**, 58–67 (1993)

Instability with Two Zero Frequencies

Z. WIENER

Department of Theoretical Mathematics,
The Weizmann Institute of Science, Rehovot 76100, Israel

Received May 8, 1990; revised April 25, 1991

In this work the instability of a degenerate equilibrium position is studied through the formal series solutions. The inversion of the Lagrange–Dirichlet stability theorem is proved in the case of two zero eigenvalues and a nondegenerate Newton's diagram. This case includes all singularities appearing in a nonremovable way in families depending on not more than 16 parameters. © 1993 Academic Press, Inc.

1. NOTATIONS

Let us consider the system of ordinary differential equations near the equilibrium position:

$$F(\xi, \dot{\xi}) = 0. \tag{1}$$

Here $\xi = (\xi^1, ..., \xi^n)^T \in \mathbb{R}^n$, $\dot{\xi} = (d/dt)\, \xi(t)$, $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is analytic, and $F(0, 0) = 0$, $F = (F^1, ..., F^n)^T$.

Choosing the quasihomogenity index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Q}^n$, $\alpha_i > 0$, defines (in the fixed coordinate system) weights (or degrees):

$$\deg[(\xi^1)^{k_1} \cdots (\xi^n)^{k_n}] = \sum_1^n \alpha_i k_i.$$

Having attached to the operator d/dt the weight 1 we can decompose F to quasihomogeneous components, using its Maclaurin series

$$F(\xi, \eta) = \sum_{j=0}^{\infty} F_j(\xi, \eta),$$

where F_j is α-quasihomogeneous vector-function of degree $\sigma + j\mu$, $\sigma = (\sigma_1, ..., \sigma_n) \in \mathbb{Q}^n$, and μ is the smallest rational number generated by 1, $\alpha_1, ..., \alpha_n$ (their greatest common divisor). In other words for every $\lambda > 0$,

$$F_j^i(\lambda^{\alpha_1}\xi^1, ..., \lambda^{\alpha_n}\xi^n; \lambda^{\alpha_1+1}\eta^1, ..., \lambda^{\alpha_n+1}\eta^n)$$
$$= \lambda^{\sigma_i + j\mu} F_j^i(\xi^1, ..., \xi^n; \eta^1, ..., \eta^n).$$

0022-0396/93 $5.00

All functions F_0^i $(1 \leqslant i \leqslant n)$ are assumed not identically zero. The vector-function F_0 is also called α-cut of (1).

Example:

$$y\dot{y}^2 + z^5 - yz^2 = 0$$
$$\dot{z} + y^2 \qquad = 0.$$

Let $\alpha = (1, 2)$; then $\deg y\dot{y}^2 = 5$, $\deg z^5 = 10$, $\deg(-yz^2) = 5$, $\deg \dot{z} = 3$, $\deg y^2 = 2$,

$$F_0 = \begin{pmatrix} y\dot{y}^2 - yz^2 \\ y^2 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 \\ \dot{z} \end{pmatrix}, \qquad F_5 = \begin{pmatrix} z^5 \\ 0 \end{pmatrix}, \qquad \sigma = (5, 2), \ \mu = 1.$$

2. Formal Solutions

We introduce the space $\mathbb{R}^n_{\ln}$ of all vector-polynomials of $\ln t$ of finite degree and consider formal series solutions of (1) in the form

$$\xi_0 t^{-\alpha} + \sum_{j=1}^{\infty} \xi_j t^{-\alpha - j\mu}, \tag{2}$$

where $\xi_j t^{-\alpha - j\mu} = (\xi_j^1 t^{-\alpha_1 - j\mu}, ..., \xi_j^n t^{-\alpha_n - j\mu})^T$ and $\xi_j \in \mathbb{R}^n$ or $\xi_j \in \mathbb{R}^n_{\ln}$, $\xi_j^i = \sum_{k=0} \xi_{jk}^i \times \ln^k t$ (this sum has only a finite number of terms). The operator d/dt corresponds to the set of linear operators on $\mathbb{R}^n_{\ln}$:

$$\mathscr{D}_j = \operatorname{diag}(-\alpha - j\mu) + \partial, \qquad \partial = \frac{d}{d \ln t}.$$

The second derivative corresponds to $C_j = \mathscr{D}_{j+\mu^{-1}} \mathscr{D}_j$.

THEOREM 1. *If for* (1) *there exists a multiindex* α *and a vector* $\xi_0 \in \mathbb{R}^n$ *such that*

1. $F_0(\xi_0, \mathscr{D}_0 \xi_0) = 0$
2. *For every* $j \geqslant 1$, $\det[(d/d\xi) F_0(\xi, \mathscr{D}_j \xi) |_{(\xi_0, \mathscr{D}_0 \xi_0)}] \neq 0$, *then there exists a formal solution in the form* (2), *where all* $\xi_j \in \mathbb{R}^n$.

Proof. Let us substitute (2) into (1). Equating terms of smallest degree gives the system $F_0(\xi, \mathscr{D}_0 \xi) = 0$; it takes place at the point ξ_0. The subsequent terms are found inductively. At the jth step all $\xi_0, ..., \xi_{j-1}$ are already known; then for ξ_j we have the system of linear algebraic equations

$$\left[\frac{d}{d\xi} F_0(\xi, \mathscr{D}_j \xi) |_{(\xi_0, \mathscr{D}_0 \xi_0)}\right] \xi_j = \zeta.$$

Vector ζ depends only on $\xi_0, \ldots, \xi_{j-1}$. Since here we consider $\xi_j \in \mathbb{R}^n$ the operator on the left side is an $n \times n$ matrix (nondegenerated by assumption 2). So this equation is solvable for every j. The proof is completed.

THEOREM 2. *If for* (1) *there exists a multiindex* α *and a vector-polynomial* $\xi_0 \in \mathbb{R}^n_{\ln}$ *such that*

1. $F_0(\xi_0, \mathscr{D}_0 \xi_0) = 0$.

2. *For every* $j \geqslant 1$, $\mathrm{Im}[(d/d\xi)\, F_0(\xi, \mathscr{D}_j \xi)|_{(\xi_0, \mathscr{D}_0 \xi_0)}] = \mathbb{R}^n_{\ln}$, *then there exists a formal solution in the form* (2), $\xi_j \in \mathbb{R}^n_{\ln}$.

THEOREM 3. *If for the system* $F(\xi, \dot{\xi}, \ddot{\xi}) = 0$ $(F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *analytic*; $F(0, 0, 0) = 0)$ *there exists* α *and* $\xi_0 \in \mathbb{R}^n_{\ln}$ *such that*

1. $F_0(\xi_0, \mathscr{D}_0 \xi_0, C_0 \xi_0) = 0$

2. *For every* $j \geqslant 1$, $\mathrm{Im}[(d/d\xi)\, F_0(\xi, \mathscr{D}_j \xi, C_j \xi)|_{(\xi_0, \mathscr{D}_0 \xi_0, C_0 \xi_0)}] = \mathbb{R}^n_{\ln}$ *then there exists a formal solution* (2), $\xi_j \in \mathbb{R}^n_{\ln}$.

Remark. If F is a sum of three terms, linear by $\ddot{\xi}$, quadratic by $\dot{\xi}$, and a term not containing time derivative (this is the case considered later), then μ can be chosen as the smallest common divisor of 2, $\alpha_1, \ldots, \alpha_n$.

The proof of Theorems 2 and 3 is completely analogous to the first proof. It is worth emphasizing that the existence of formal solution is claimed here (the series can diverge). The connection between formal solution and the actual one is given by the following Kuznetsov's theorem [1].

THEOREM. *If for* (1) *there exists a formal solution of the type* (2) *with* $\xi_j \in \mathbb{R}^n_{\ln}$ *then there exists an actual solution such that it has the formal one as an asymptotic expansion.*

3. INSTABILITY PROBLEM

Consider a mechanical system in a potential field of external forces near the equilibrium position which is not a local energy minimum. The case when the quadratic part of the potential is nonnegative but degenerate is considered. Under some assumptions instability of such equilibrium is proved. We find a formal asymptotic solution in the form (2). It is called asymptotic since each term tends to zero when time goes to infinity. According to Kuznetsov's theorem an actual solution exists and has the same asymptotic expansion. Using invertability of time we get instability immediately.

Applying the splitting lemma [2] in the normal system of coordinates [3] we can write the system of Euler–Lagrange equations in the form

$$\frac{\partial L}{\partial \xi}=\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}}. \tag{3}$$

Here $\xi \in \mathbb{R}^n$, vector from the configuration space, the Lagrangian takes the form

$$L(\xi, \dot{\xi}) = T(\xi, \dot{\xi}) - V(\xi^1, ..., \xi^l) - \frac{1}{2}\sum_{i=l+1}^{n} (\xi^i)^2.$$

Here T stands for the kinetic energy, being quadratic by the velocities, and positively defined for every position ξ, also $T(0, \dot{\xi}) = \frac{1}{2}\sum_1^n (\dot{\xi}^i)^2$. The nondegenerate quadratic part of the potential is $\frac{1}{2}\sum_{l+1}^n (\xi^i)^2$; to simplify calculations we assume all eigenvalues equal to one. The rest of the potential is V; its Maclaurin expansion starts with terms of degree at least three and the origin is not a local minimum of V. So, the coordinate form of (3) is

$$\begin{aligned} \ddot{\xi}^i &= -\frac{\partial}{\partial \xi^i} V(\xi^1, ..., \xi^l) + K(\xi, \dot{\xi}) + \mathcal{O}(\xi)\,\ddot{\xi}, && 1 \leqslant i \leqslant l, \\ \ddot{\xi}^i &= -\xi^i + K(\xi, \dot{\xi}) + \mathcal{O}(\xi)\ddot{\xi}, && l+1 \leqslant i \leqslant n. \end{aligned} \tag{4}$$

Here and later $K(\cdot, \cdot)$ stands for any (different in different formulas) analytic function, quadratic in the second argument, and $\mathcal{O}(\xi)$ is a linear operator such that $\|\mathcal{O}(\xi)\| \leqslant c\,|\xi|$ for some constant c and small $|\xi|$.

Further we consider the case $l=2$ and to simplify formulas we also assume $n=3$, but the same reasoning is true for any n. We denote ξ^1 by x, ξ^2 by z, and ξ^3 by p.

Let us study Newton's diagram of the function V (all necessary definitions are in [4]). Since the origin is not a local minimum of potential energy, there are nonpositive[1] edges of the diagram [5]. Our first assumption is that there exists at least one negative edge (of any dimension). The nearest to the homogeneous form negative edge we denote Γ and function generated by Γ by W. The case when Γ is a homogeneous form is studied in [3], so the quasihomogeneity index is (β_1, β_2) ($\beta_1 \neq \beta_2$, say $\beta_1 < \beta_2$); it is a vector with rational positive components, orthogonal to Γ. For any $\lambda \geqslant 0$,

$$W(\lambda^{\beta_1}x, \lambda^{\beta_2}z) = \lambda^s W(x, z);$$

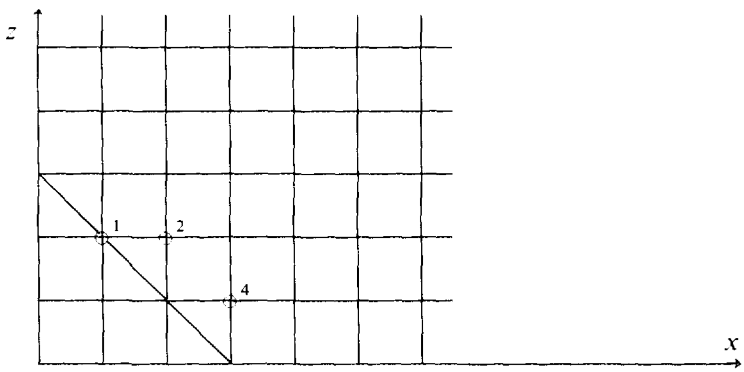
[1] A cluster of monomials is called positive if the function defined by them is strictly positive everyhwere near the origin outside the coordinate surfaces ($\exists \varepsilon > 0 : \forall\, |\xi| < \varepsilon;\ \xi^i \neq 0,\ 1 \leqslant i \leqslant n \Rightarrow f(\xi) > 0$). A cluster of monomials is called negative if the function defined by them can have negative values in any neighbourhood of the origin. The cluster is called degenerated if it concerns neither of these two classes.

$s \in \mathbb{Q}$ is a β-quasidegree of W. Now the function V is decomposed to quasihomogeneous components:

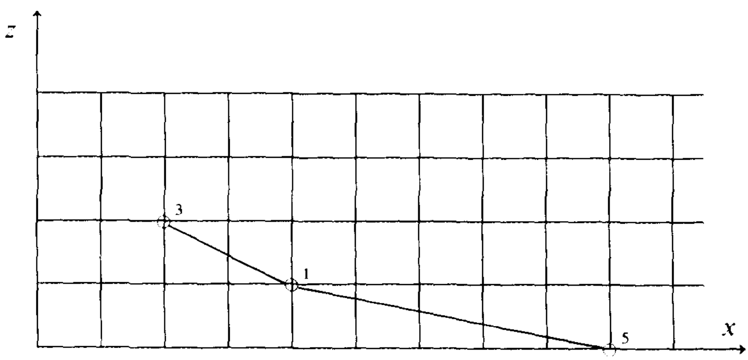
$$V(x, z) = W(x, z) + \sum_{j=1}^{k} V_j(x, z)$$
$$V_j(\lambda^{\beta_1} x, \lambda^{\beta_2} z) = \lambda^{s + j\mu} V_j(x, z).$$

Since β is defined up to multiplication by some constant, one can choose it so that $s - \beta_1 = 2 + \beta_1$. As a multiindex from Theorem 3 we take, $\alpha \in \mathbb{Q}^3$, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_1$. As will be seen later, it is possible to choose any α_3 greater than or equal to β_1, less than or equal to $2 + 2\beta_1$, since the first nontrivial term in the $p(t)$ expansion is of the type t^{-k}, $k \geqslant 2 + 2\beta_1$.

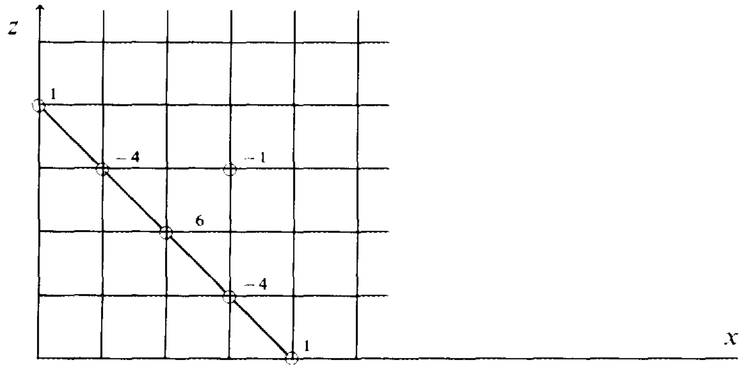
EXAMPLES. 1. The function $V(x, z) = xz^2 + 2x^2z^2 + 4x^3z$ has a negative homogeneous zero-dimensional edge (case studied in [3]). Here $\beta_1 = \beta_2$, $W(x, z) = xz^2$.



2. The function $V(x, z) = 3x^2z^2 + x^4z + 5x^9$ satisfies our conditions, because its one-dimensional $(1, 2)$-quasihomogeneous edge Γ is negative. Here $W(x, z) = 3x^2z^2 + x^4z$.



3. The function $V(x, z) = (x - z)^4 - x^3 z^3$ does not satisfy the assumption because its Newton diagram has only positive (zero-dimensional) and degenerate (one-dimensional) edges.



In the new notations Eqs. (4) read for $\xi = (x, y, p)^T$,

$$\begin{aligned}
\ddot{x} &= -\frac{\partial W}{\partial x}(x, z) - \sum_{j=1}^{\infty} \frac{\partial}{\partial x} V_j(x, z) + K(\xi, \dot{\xi}) + \mathcal{O}(\xi)\ddot{\xi} \\
\ddot{z} &= -\frac{\partial W}{\partial z}(x, z) - \sum_{j=1}^{\infty} \frac{\partial}{\partial z} V_j(x, z) + K(\xi, \dot{\xi}) + \mathcal{O}(\xi)\ddot{\xi} \qquad (5)\\
\ddot{p} &= -p + K(\xi, \dot{\xi}) + \mathcal{O}(\xi)\ddot{\xi}.
\end{aligned}$$

To find its α-cut we build the table of degrees: $\deg \ddot{x} = s - \alpha_1 = 2 + \alpha_1$; $\deg(\partial/\partial x) W = s - \alpha_1$; $\deg(\partial/\partial x) V_j = s - \alpha_1 + \mu j$; $\deg K \geqslant 2 + 2\alpha_1$; $\deg \mathcal{O}(\xi)\ddot{\xi} \geqslant 2 + 2\alpha_1$; $\deg \ddot{z} = 2 + \alpha_2$; $\deg(\partial/\partial z) W = s - \alpha_2$; $\deg \ddot{p} = 2 + \alpha_3 = 2 + \alpha_1$; $\deg p = \alpha_3 = \alpha_1$. So the α-cut is

$$\begin{aligned}
\ddot{x} &= -\frac{\partial}{\partial x} W(x, z) \\
0 &= -\frac{\partial}{\partial z} W(x, z) \qquad (6)\\
0 &= -p.
\end{aligned}$$

Consider the function W on the cylinder $|x| = 1$. There exists a point $e = (e^1, e^2)^T$ such that

1. $W(e) < 0$
2. $(\partial/\partial x)\, W(e) \cdot e^1 < 0$, $|e^1| = 1$
3. $(\partial/\partial z)\, W(e) = 0$
4. $(\partial^2/\partial z^2)\, W(e) \geqslant 0$.

It follows from the conditions that the left vertex of Γ is a positive edge; hence $\lim_{|z| \to \infty} W(e^1, z) = +\infty$.

Our second assumption is that $(\partial^2/\partial z^2)\, W(e) > 0$.

Let us choose an initial vector ξ_0 in the form $\xi_0 = (\lambda^{\alpha_1} e^1, \lambda^{\alpha_2} e^2, 0)$. The first condition of Theorem 3 has the form

$$\alpha_1(\alpha_1 + 1)\, \lambda^{\alpha_1} e_1 = -\lambda^{s-\alpha_1} \frac{\partial}{\partial x} W(e)$$

$$0 = -\lambda^{s-\alpha_2} \frac{\partial}{\partial x} W(e)$$

$$0 = 0.$$

It is satisfied by choosing

$$\lambda = \left[\frac{-\alpha_1(\alpha_1 + 1)\, e^1}{(\partial W/\partial x)(e)} \right]^{1/(s - 2\alpha_1)}.$$

The operator from the second condition of Theorem 3 has the form

$$\begin{pmatrix} \lambda^{s-2\alpha_1} \cdot (\partial^2/\partial x^2)\, W(e) & \lambda^{s-\alpha_1-\alpha_2}(\partial^2/\partial x\, \partial z)\, W(e) & 0 \\ \lambda^{s-\alpha_1-\alpha_2}(\partial^2/\partial z\, \partial x)\, W(e) & \lambda^{s-2\alpha_2}(\partial^2/\partial z^2)\, W(e) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} c_j$$

$$= \begin{pmatrix} \lambda^{s-2\alpha_1}(\partial^2/\partial x^2)\, W(e) + c_j & \lambda^{s-\alpha_1-\alpha_2}(\partial^2/\partial x\, \partial z)\, W(e) & 0 \\ \lambda^{s-\alpha_1-\alpha_2}(\partial^2/\partial x\, \partial z)\, W(e) & \lambda^{s-2\alpha_2}(\partial^2/\partial z^2)\, W(e) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} \bar{c}_j \partial + \partial^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = B_j + \bar{B}_j,$$

Here $c_j = (\alpha_1 + \mu j)(\alpha_1 + \mu j + 1)$; $\bar{c}_j = 2\alpha_1 + 2\mu_j + 1$. The operator $B_j + \bar{B}_j$ maps $\mathbb{R}^n_{\text{In}}$ to $\mathbb{R}^n_{\text{In}}$. Observe that if $\det B_j \neq 0$ then $\operatorname{Im}(B_j + \bar{B}_j) = \mathbb{R}^n_{\text{In}}$. But $\det B_j$ depends linearly on c_j, which is monotonically increasing in j. So there exists not more than one critical number j^0 such that $\det B_{j^0} = 0$.

Straightforward calculations show that the critical value is $j^0 = \mu^{-1}$. If it is not a natural number then the determinant is never equal to zero and the formal solution can be found in the form (2) with $\xi_j \in \mathbb{R}^n$ (without ln). It takes place for example when s is an odd number. Otherwise the following propositions are true:

1. $\dim \operatorname{Ker} B_{j0} = 1$ and $(1, z, 0)^T \in \operatorname{Ker} B_{j0}$ for some z.
2. $(1, 0, 0)^T \notin \operatorname{Im} B_{j0}$.

Hence $\operatorname{Im} B_{j0} + \overline{B_{j0}}(\operatorname{Ker} B_{j0}) = \mathbb{R}^n_{\ln}$. More details can be found in [6]. All conditions of Theorem 3 are checked. So we have proved the existence of a formal solution. Thanks to Kuznetsov's theorem there exists also an actual asymptotic solution. Since time is invertible in such systems, the instability of equilibrium is proved. Now we can formulate.

THEOREM 4. *The equilibrium position with the following three conditions is unstable.*

1. *There are not more than two zero eigenfrequencies.*
2. *The absence of the local minimum at the equilibrium position can be established through Newton's diagram.*
3. *The "direction of the fastest descent of the potential" is not degenerated (i.e., $(\partial^2 W/\partial z^2)(e) \neq 0$).*

4. APPENDIX

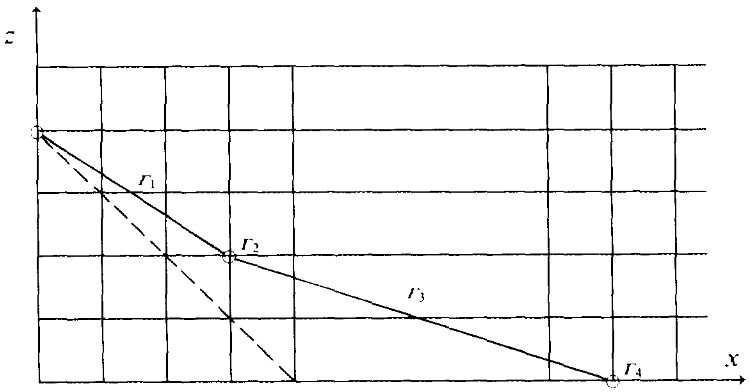
For degenerate critical points of smooth real functions of many variables the normal forms (which are unremovable in smooth families with not too great a number of parameters) have been computed [4].

THEOREM 5. *If the equilibrium position is a simple or uni- or bimodal singularity of the potential, with no local minimum at this point, then the motion equations have a solution, asymptotically approaching this equilibrium position.*

In the classes of functions of codimension not greater than 16, only these types of singularities can appear in an unremovable way. For simple and unimodal cases this theorem has been proved in [3]. The proof in bimodal cases uses the table of normal forms of germs of smooth functions, composed by Zakalukin. All the forms can be analyzed through Newton's diagram since they have either cubic terms or not more than two zero eigenfrequencies.

EXAMPLE. To demonstrate the work of the algorithm let us apply it to the analysis of the singularity of the $W_{1,k}$ type (of [4]). The corresponding potential is $V = z^4 + z^2x^3 + x^6(a+bx)x^k + (p^3)^2 + \cdots + (p^n)^2$.

Newton's diagram has the form



The first nontrivial form is the fourth one (dotted line). The corresponding edge consists of the only monomial z^4 and is positive. So the results of [3] cannot be used. The four edges $\Gamma_1 = \{z^4, z^2x^3\}$, $\Gamma_2 = \{z^2x^3\}$, $\Gamma_3 = \{z^2x^3, ax^{6+k}\}$, $\Gamma_4 = \{ax^{6+k}\}$ (when a is negative or k odd) are negative. We must choose, however, only Γ_1 for only its left end is a positive edge. The corresponding indices are $\alpha_1 = \frac{2}{4}$, $\alpha_2 = \frac{3}{4}$; $s = 3$. After finding these numbers the following steps are simple. To shorten all the computations let us assume $a = 1$, $b = 0$, $n = 3$, $\mathcal{O}(\xi)\ddot{\xi} = 0$, $K(\xi, \dot{\xi}) = (0, -\dot{x}^2, -\dot{x}^2)^T$.

The system (5) has the form

$$\ddot{x} = -3z^2x^2 - 7x^6$$
$$\ddot{z} = -4x^3 - 2zx^3 + \dot{x}^2$$
$$\ddot{p} = -p + \dot{x}^2.$$

The α-cut (6) after substituting the series is given by

$$c_0x_0 = -3z_0^2x_0^2$$
$$0 = -4z_0^3 - 2z_0x_0^3$$
$$0 = -p_0.$$

Its solution x_0, z_0, p_0 can be easily obtained. Afterward, at each step only a linear system with nondegenerate (when $\mu^{-1} \neq j$) matrix B_j will appear.

At the fourth ($\mu = \frac{1}{4}$) step $\det B_4 = 0$. Thus, starting from this step $\xi_j(\cdot)$ is no longer a vector but rather a vector-polynomial of $\ln t$. In such a way the whole formal solution can be constructed.

Acknowledgments

The author is thankful to V. V. Kozlov for his constant encouragement and to V. A. Vasiljev for useful discussions.

References

1. A. N. Kuznetsov, On existence of asymptotic solution of an autonomous system with formal solution, *Functional Anal. Appl.* **23**, No. 4 (1989), 63–74.
2. T. Poston and I. Stewart, "Catastrophe Theory and Its Applications," Pitman, New York, 1978.
3. V. V. Kozlov, Asymptotical motions and the problem of inversion of Lagrange–Dirichlet theorem, *Prikl. Math. Mech.* **50**, No. 6 (1986), 928–937.
4. V. I. Arnold, A. N. Varchenko, and S. M. Gusein-Zade, "Singularities of Differentiable Maps," Vol. 1, Monographs in Mathematics, Boston, 1985.
5. V. A. Vasiljev, Asymptotic exponential integrals, Newton's diagram, and the classification of minimal points, *Functional Anal. Appl.* **11**, No. 3 (1978), 163–172.
6. G. M. Vinner (Z. Wiener), Asymptotical motions of mechanical systems with nonholonomic constraints, *Prikl. Math. Mech.* **53**, No. 4 (1989), 549–553.