The Analysis of Deltas, State Prices and VaR: A New Approach

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We provide a monotonic transformation of an initial diffusion with a level-dependent volatility parameter that yields a second, deterministic diffusion parameter process. Limited information about the volatility parameter of the initial process can bound the drift of the transformed process so that probabilities under the initial diffusion can be bounded in terms of probabilities under arithmetic Brownian motion. These probability bounds provide new theoretical bounds on deltas, state prices and VaR applicable when one has only limited information about the underlying's volatility. When an asset's volatility depends on its price, deltas and state prices depend on nominal prices, nominal interest rates *and* the inflation rate.

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Suppose an asset's price follows a one-dimensional diffusion. Investors in that asset and derivatives thereon face three related problems. The first concerns determining the Value at Risk (VaR) in the event of a price collapse with, say, a 1-in-20 chance of occurrence.[1] The VaR, determined by the *true* price process, must often be estimated given only coarse knowledge of the complex relation between the asset's volatility and its value. The valuation of contingent claims requires the determination of the probability distribution of the asset's future returns under the *risk-neutral* process. Although the drift of the risk-neutral process is known from the risk-free rate, limited knowledge of the functional form of the volatility parameter of the true price process implies only limited knowledge of that same parameter of the risk-neutral process. Hedging that contingent claim requires the calculation of position deltas. Deltas are determined by the probability distribution of the underlying asset's future returns under the *delta* process. Limited knowledge of the functional form of the volatility parameter of the true price process implies only limited knowledge of both the drift and the diffusion parameters of the delta process. Thus all three problems involve the determination of probabilities of events given only limited knowledge of the relevant diffusion.

This article shows how to use limited knowledge of the relation between an asset's volatility and its price to bound the VaR associated with investing in the asset and bound the prices and deltas of derivative claims thereon. The bounds we derive follow from two observations. First, the probabilities of interest can be determined as the probabilities of the equivalent events under any monotonic transformation of the original process. Second, in the setting considered in this article, one can always monotonically transform an original diffusion process with a level-dependent diffusion parameter into a second diffusion process with a deterministic diffusion parameter. While the increased tractability due to a deterministic diffusion parameter comes at the expense of a more complex drift parameter, limited knowledge of the functional form of the volatility parameter of the

[1] See Linsmeier and Pearson (1996) and Duffie and Pan (1997) for reviews of techniques in this area.

original process can be used to place a deterministic bound on the drift of the transformed process. Thus bounds on probabilities can be determined by calculating the probabilities of related events under an arithmetic Brownian motion process, these bounds take the familiar form of cumulative standardized normal density functions.

The Merton (1973) classic no-arbitrage bounds on option prices presume no knowledge of the volatility parameter. The similarly "weak" bounds on contingent claims' deltas given in Theorem 1 of Bergman, Grundy and Wiener (1996) presume only that the underlying asset's price follows a diffusion process. Complete knowledge of that diffusion's volatility parameter would allow one to price and hedge options exactly. This article considers a middle ground—what can one say about pricing and hedging given some limited knowledge of the volatility parameter. Limited knowledge could, in principal, take a myriad of forms. We determine the relevant bounds on deltas, state prices and VaR when one knows, or can bound, volatility at the asset's current price level and/or some alternate price level and, in addition, that the volatility parameter satisfies one or more of the following restrictions: (*i*) the elasticity of volatility with respect to the asset's price is at least -1; (*ii*) the volatility is non-decreasing in the asset's price; (*iii*) the volatility is non-increasing in the asset's price; and (*iv*) the volatility is everywhere bounded above and/or below.

In order to derive bounds on VaR, options prices, and option deltas, it is first necessary to characterize the source of any time-dependence in the volatility parameter. There are two natural sources of time-dependence: time variation in the arrival rate of information concerning the underlying asset; and inflation. While the first source is obvious, the link between time-dependence and inflation is less familiar. Whenever an asset's volatility depends on its price, that dependence will be on the real, and not the nominal, price. Thus when the diffusion describing changes in an asset's value is written in terms of nominal prices, the volatility parameter thereof must depend on both the nominal price and, in order to reflect the appropriate inflation adjustment, time as well. The time-dependence in an asset's volatility parameter implies a natural choice from the large set of functions that monotonically transform a general one-dimensional diffusion process into a process with a deterministic diffusion parameter. Distinguishing between real and nominal prices allows us to establish that when restriction (i) is satisfied and the real interest rate is non-negative, then the delta of a call option is at least 1/2 whenever the underlying asset's real price exceeds the real value of the option's exercise price. We also demonstrate that this result need not be true when the volatility parameter of the price process does not satisfy restriction (i). We do so by deriving an analytic solution to a particular call option pricing problem in which the underlying asset's diffusion parameter is first increasing and then, in violation of restriction (i), decreasing in the price of the underlying asset. For this particular process, the delta of an at-the-money call is, in direct contrast to the result in a Black-Scholes setting, always *less* than 1/2.

Our work on analyzing probabilities by considering the equivalent event under a monotonic transformation of the original diffusion process is an extension of results in two earlier papers. Nelson and Ramaswamy (1990) use a specific case of the transformation used in this article to obtain the basis of their development of a computationally simple binomial approximation to a diffusion process with a level-dependent volatility. The approach in our article is similar in spirit to that of Goldenberg (1991). Goldenberg shows that when an initial one-dimensional risk-neutral diffusion process can be transformed to obtain a second process with a known transition density function, closed-form option pricing models can be determined from that known transition density. Goldenberg uses this result to establish two new classes of closed-form option pricing models: The class of residual volatility option pricing models corresponds to the set of risk-neutral processes that can be transformed to yield arithmetic Brownian motion; and the class of generalized square root option pricing models corresponds to the set of risk-neutral processes that can be transformed to yield a standard Bessel diffusion. Unlike Goldenberg, our interest is in the case where the absence of complete knowledge of the volatility parameter of the original process precludes, by default, a transformation to a process with a known transition density. We show that limited knowledge of the volatility parameter of the original process can still allow a transformation to a second process whose transition density function can be bounded in terms of the transition density of an arithmetic Brownian motion process.

Section 1 contains the analysis of probabilities in diffusion settings and establishes the set of transformations that yield a deterministic diffusion parameter process. The economics of time-dependence in an asset's volatility parameter are used to select a particular transformation from that set that is applied throughout the remainder of the article. Section 2 defines the delta process and shows that a call option's delta is equivalent to its probability of finishing in-the-money under the delta process. Bounds on deltas for broad classes of underlying stock price processes are contained in Lemma 1. Tables are used to numerically illustrate the tightness of these bounds on true deltas when the underlying asset follows a CEV process and when the underlying follows a displaced diffusion process. Bounds on state prices are derived in Section 3 and set out in Lemma 2. Provided that restriction (i) is satisfied, we show that deltas can be bounded in terms of state prices, and hence that the information about state prices implicit in even a coarse grid of observed option prices can be used to place an empirical bound on deltas. Section 4 shows how to use limited knowledge of the functional form of the volatility parameter to place bounds on the VaR associated with investing in the underlying asset and provides a numerical illustration of these bounds. Bounds on true probabilities and VaR are contained in Lemma 3 and 4 respectively. Section 5 contains our conclusions and suggests extensions of this line of research.

1. Some Properties of Probabilities in Diffusion Settings

Let $\xi_\tau^{y,t}$ denote the time τ value of a diffusion that at time $t < \tau$ starts at the level y and then obeys the stochastic differential equation (SDE)

$$\begin{aligned} d\xi_\tau &= \mu(\xi_\tau, \tau)d\tau + \sigma(\xi_\tau, \tau)\xi_\tau dB_\tau \\ &= \mu(\xi_\tau, \tau)d\tau + z(\xi_\tau, \tau)dB_\tau. \end{aligned} \tag{1}$$

We follow the finance literature and refer to the function $\sigma(\cdot)$ as the *volatility* parameter. Following Karlin and Taylor (1981, p. 159) we refer to the product $z(\cdot) := \sigma(\cdot)\xi$ as the *diffusion* parameter. We use numerical subscripts to denote partial derivatives. Thus, for example $z_{11}(\xi, t)$ is the second partial of the diffusion parameter with respect to its first argument, the level of the process. In addition to imposing Lipschitz and growth restrictions on the parameters μ and z that assure the existence of a unique Ito process satisfying (1) for each possible starting value in $\mathbb{R}^+$, whenever we apply Theorem 1 (developed below) we will also assume that σ satisfies:

Assumptions Set. (i) σ is once differentiable in ξ and once differentiable in t. (ii) $\sigma(\xi, t) > 0$ for all $\xi > 0$ and all t. (iii) The function F given (below) in expression (2) is well defined for all $\xi \geq 0$.

Consider the probability of the event that at time T the process has reached a level above k; $\Pr\left(\xi_T^{y,t} > k\right)$. The analysis of this probability is complicated by the dependence of the volatility parameter on the level of the process. Hence our interest in transforming the original problem into one involving a process whose diffusion parameter takes a particularly simple form. We first note that the probability of interest is equal to the probability of the following equivalent event.

Observation 1. **(Preservation of Probability)** *Suppose ξ starts at y and follows the diffusion in* (1)*. If the function $F(\xi, t)$ is strictly increasing in ξ, then*

$$\Pr\left(\xi_T^{y,t} > k\right) = \Pr\left(F(\xi_T^{y,t}, T) > F(k, T)\right).$$

When F is twice differentiable in ξ and once differentiable in t, the random variable $F(\xi, t)$ will follow a diffusion. Consider one such specification of the F function

$$F(\xi, t) := \int_{A(t)}^{\xi\alpha(t)} \frac{a(t)}{\sigma(x/\alpha(t), t)x} dx, \tag{2}$$

where $a(t)$, $A(t)$ and $\alpha(t)$ are positive smooth functions of time. For this specification of the F function, the transformed process has a deterministic diffusion parameter. A similar form of simplifying transformation was used for reducing state dependence of volatility and drift by Ricciardi (1976). In financial literature similar transformation has been used by Jamshidian (1991) and Goldenberg (1991). In order to apply this transformation one has to know σ exactly, since it appears in the denominator, however as we show below in some cases even qualitative information about σ can be used for getting bounds on some economic variables.

Observation 2. **(Transformation to a Deterministic Diffusion Parameter Process)** *Suppose ξ starts at y at time t and follows the diffusion in* (1)*, and the restrictions of the Assumption Set are satisfied. For the function $F(\xi, t)$ defined in* (2)*, the dynamic of F is given by*

$$dF_\tau = \psi(F_\tau, \tau)d\tau + a(\tau)dB_\tau,$$

where the functional form of the drift parameter ψ is set out in Appendix A.

The straightforward application of Ito's Lemma that underlies Observation 2 is set out in Appendix A. The transformation in (2) is a generalization of the familiar technique of using the natural log to transform a process following geometric Brownian motion into a process following arithmetic Brownian motion. Rather than transforming the drift while maintaining the diffusion parameter at the expense of a change in the probability measure as in Girsanov's Theorem, one can transform the diffusion parameter while maintaining the Wiener process at the expense of a change in the drift. The functions $a(t)$, $A(t)$ and $\alpha(t)$ in (2) are choices to be made in defining F. The choice of the function $a(t)$ is, effectively, the choice of the deterministic diffusion parameter in the dynamic of F. Given the choice of the $\alpha(t)$ function, the choice of the function $A(t)$ can be thought of as the choice of the time t starting value of the F process. The choice of the $\alpha(t)$ function influences the drift of the F process and, as will be seen, can be critical in analyzing certain interesting economic settings. In particular, when the diffusion process describes the dynamic of an asset's price and that asset's volatility is a function of its price, then the appropriate choice of $\alpha(t)$ will prove useful in appropriately distinguishing its real price and its nominal price. One can also consider an alternate transformation of the original process provided by the function $\mathcal{F}$ with the property that the diffusion parameter of the dynamic of $\mathcal{F}$ is directly proportional to $\mathcal{F}$; i.e., the volatility parameter of the dynamic of $\mathcal{F}$ is deterministic.

Observation 2$'$. (Transformation to a Deterministic Volatility Parameter Process) *Suppose ξ starts at y at time t and follows the diffusion in (1), and the restrictions of the Assumption Set are satisfied. For the function $\mathcal{F}$ defined in*

$$\mathcal{F}(\xi, t) := \exp\left(\int_{A(t)}^{\xi\alpha(t)} \frac{a(t)}{\sigma(x/\alpha(t), t)x} dx \right),$$

the dynamic of $\mathcal{F}$ is given by

$$d\mathcal{F}_\tau = \lambda(\mathcal{F}_\tau, \tau)d\tau + a(\tau)\mathcal{F}_\tau dB_\tau,$$

where the functional form of the drift parameter λ is set out in Appendix A.

The F function of Observation 2 can be used along with Observation 1 to develop a new expression for $\Pr\left(\xi_T^{y,t} > k\right)$ that we state as Theorem 1. This new expression is an expression for the probability of the equivalent event that $F(\xi_T^{y,t}, T) > F(k, T)$.

Theorem 1. *Suppose ξ starts at y at time t and follows the diffusion in (1), and the restrictions of the Assumption Set are satisfied. Let $a(t)$, $A(t)$ and $\alpha(t)$ be positive smooth functions of t. Define the function F as in (2) and let $\psi(F_\tau, \tau)$ denote the drift parameter of the dynamic of F. The probability that $\xi_T^{y,t}$ exceeds k can be expressed as*

$$\Pr\left(\xi_T^{y,t} > k\right) = \Pr\left(\frac{F(y,t) - F(k,T) + \int_t^T \psi(F_\tau, \tau)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right) \tag{3}$$

where the random variable $\mathcal{X} := -\frac{\int_t^T a(\tau)dB_\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}$ is distributed $\mathcal{N}(0, 1)$.

Theorem 1 is proved in Appendix A.

If one had complete knowledge of the functional form of the volatility parameter, one could then use Theorem 1 to determine the relevant probability exactly. Our interest is in settings where one has only limited knowledge of $\sigma(\xi, t)$, yet that limited knowledge allows one to determine lower and upper bounds on both $\psi(F(\xi, t), t)$ and $F(y, t) - F(k, T)$ such that, for all ξ and t,

$$l(t) \leq \psi(F(\xi, t), t) \leq u(t),$$

and, for all y and t,

$$L(y, t) \leq F(y, t) - F(k, T) \leq U(y, t).$$

Given a deterministic bound on the drift of the F process, Proposition 2.18 of Chapter 5 of Karatzas and Shreve (1991) implies a first-order stochastic dominance relation between the distribution of random increments to F and the distribution of random increments over the same time interval in an arithmetic Brownian motion process with a unit diffusion parameter and a drift parameter equal to the deterministic bound.

Whenever our limited knowledge of the volatility parameter allows us to determine $l(t)$ and

$L(y,t)$ and/or $u(t)$ and $U(y,t)$, Theorem 1 then implies

$$\Pr\left(\xi_T^{y,t} > k\right) \geq \Pr\left(\frac{L(y,t)+\int_t^T l(\tau)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right) \tag{4a}$$

$$= N\left(\frac{L(y,t)+\int_t^T l(\tau)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}\right), \tag{4b}$$

and

$$\Pr\left(\xi_T^{y,t} > k\right) \leq \Pr\left(\frac{U(y,t)+\int_t^T u(\tau)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right) \tag{5a}$$

$$= N\left(\frac{U(y,t)+\int_t^T u(\tau)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}\right). \tag{5b}$$

Even if one had complete knowledge of the functional form of the volatility parameter, and hence could work directly with Theorem 1, the problem would be difficult since the random variables on each side of the inequality in (3) are not, in general, independent. But when our limited information allows us to place a *deterministic* bound on $\psi(F(\xi,t),t)$, then the left-hand-sides of the inequalities in $(4a)$ and $(5a)$ are not random variables, and hence the simplification to $(4b)$ and $(5b)$. Consider the problem of placing a deterministic bound on $\psi(F(\xi,t),t)$. As shown in Appendix A,

$$\psi(F(\xi,t),t) = a(t)\left(\frac{\mu(\xi,t)}{z(\xi,t)} - \frac{1}{2}z_1(\xi,t)\right) + F_2(\xi,t). \tag{6}$$

One determinant of $F_2(\xi,t)$ is $\sigma_2(\xi,t)$, and hence placing a bound on ψ requires an understanding of the economics of any time-dependence in the volatility parameter. As we now show, the nature of this time-dependence implies natural choices for the functions $a(t)$, $A(t)$, and $\alpha(t)$, choices which allow the F_2 component of ψ to be bounded.

In what follows we consider several examples of how this result can be used. One is when volatility is multiplicatively separable, then the transformation (2) takes simple form. Another case is when the time dependence is due to inflation only and also leads to significant simplifications. However we should notice that these are only examples one can not derive a general class of processes when the transformation becomes simple. The existsing results provide a necessary and sufficient conditions for a transformation that makes both drift and volatility state independent, see Ricciardi (1976) and Schuss (1980), our goal is different since we do not have to make the drift state independent.

1.1. The economics of a time-dependent volatility parameter

Let s denote the nominal price of an asset whose risk-neutral dynamic is given by:

$$\begin{aligned} d\xi_\tau &= r(\tau)\xi_\tau d\tau + \sigma(\xi_\tau, \tau)\xi_\tau dB_\tau \\ &= r(\tau)\xi_\tau d\tau + z(\xi_\tau, \tau) dB_\tau, \end{aligned} \tag{7}$$

where $r(t)$ is the time t nominal risk-free rate. A dependence of $\sigma(\cdot)$ on τ is often viewed as an innocuous generalization of the volatility parameter to allow for time-varying information arrival. There is a second and more important reason to work with a time-dependent $\sigma(\cdot)$ function. Whenever $\sigma(\cdot)$ depends on s and inflation is non-zero, then the volatility parameter must depend on both s *and* time. The implications for option pricing are made clear in section 2.1.

1.1.A. Time-varying information arrival

The volatility of the price process is time-dependent when more information about the underlying asset is released at some times than at other times—see Patell and Wolfson (1979) and (1981). This is captured by modeling the volatility parameter as multiplicatively separable in its time- and price-dependence.[2]

$$\sigma(s, t) := m(t)n(s).$$

For $a(t) = m(t)$, $A(t) = k$ and $\alpha(t) = 1$, the function F takes the form

$$F(\xi, t) = \int_k^\xi \frac{m(t)}{m(t)n(x)x} dx = \int_k^\xi \frac{1}{n(x)x} dx,$$

with the immediate property that for all ξ and t,

$$F_2(\xi, t) = 0.$$

1.1.B. Inflation and a price-dependent volatility parameter

[2] This simple form includes two interesting special cases: volatility does not depend on time when $m(t) \equiv 1$; and volatility is a deterministic function of time when $n(s) \equiv 1$ (a Black-Scholes setting).

For notational ease only, we assume henceforth that $r(t) = r$ for all t. For similar notational ease we also assume that the deterministic inflation rate is a constant, i. The real risk-free rate is then the constant $\phi = r - i$. Assume for the moment that information about the underlying asset does arrive uniformly through time. Let q denote the real value of a stock. The risk-neutral process describing changes in the *real* value of the stock is assumed to be:

$$dq_\tau = \phi q_\tau d\tau + v(q_\tau) q_\tau dB_\tau \,.$$

Note that the volatility function $v(q)$ is *not* time-dependent. Let p_t denote the price level at time t, with the price level normalized to unity at time 0. Let $s_t = p_t \times q_t = e^{it} \times q_t$ denote the time t nominal price of the stock. From Ito's Lemma we have that the nominal price of the stock follows the diffusion:

$$\begin{aligned} ds_\tau &= (\phi + i) s_\tau \, d\tau + v(q_\tau) s_\tau \, dB_\tau \\ &= (\phi + i) s_\tau \, d\tau + v(e^{-i\tau} s_\tau) s_\tau \, dB_\tau \\ &= r s_\tau \, d\tau + \sigma(s_\tau, \tau) s_\tau \, dB_\tau \,. \end{aligned}$$

Even though the volatility function v is not time-dependent, the volatility function σ is time-dependent and, in this case, the volatility parameter in (7) will take the form:

$$\sigma(s, t) = v(se^{-it}).$$

As a further example of the need for care when working with a price-dependent volatility function in a world of non-zero inflation, suppose that the risk-neutral process for the *nominal* price of a stock is given by:

$$ds_\tau = r s_\tau d\tau + \sigma(s_\tau) s_\tau dB_\tau \,.$$

Note that the function $\sigma(s)$ is *not* time-dependent. This is a very common specification of the risk-neutral process. But this specification is far from innocuous. The assumed lack of time-dependence in the volatility function $\sigma(\cdot)$ has the non-trivial economic implication that the constant inflation rate i must be $i = 0$!

In general, the time-dependence of volatility can reflect both a non-zero inflation rate and a time varying rate of information release:

$$\sigma(s,t) = m(t)v(se^{-it}).$$

Without loss of generality, we assume in the remainder of the article that $m(t) = 1$ for all t. In defining F we always choose $a(t) = 1$, $A(t) = k$ (later it will be used as a strike price) and $\alpha(t) = e^{i(T-t)}$.[3]

$$\begin{aligned} F(\xi,t) &= \int_k^{\xi e^{i(T-t)}} \frac{1}{\sigma(xe^{-i(T-t)},t)x}dx \\ &= \int_k^{\xi e^{i(T-t)}} \frac{1}{v(xe^{-iT})x}dx, \end{aligned}$$

and

$$F_2(\xi,t) = -\frac{i\xi e^{i(T-t)}}{v(\xi e^{i(T-t)}e^{iT})\xi e^{i(T-t)}} = -\frac{i}{\sigma(\xi,t)}.$$

Armed with Theorem 1 and this specification of the F function we now turn to the analysis of the problems of hedging and pricing derivatives and the determination of VaR when we have only limited knowledge of the volatility parameter of the relevant diffusion.

2. Bounds on Deltas

In order to apply Theorem 1 to the analysis of position deltas, we must first demonstrate that deltas are determined by the probability distribution of the realization of a diffusion process at the position's terminal date. Let $V(s,t)$ denote the nominal price of a European contingent claim on an underlying asset whose risk-neutral process is given by (7). When, in particular, a call option is considered, c instead of V will be used to denote its nominal price. The contractual payoff function is $g(\cdot)$, meaning that if the underlying price is s at the expiration date T, then the contingent claim will pay off $g(s)$. Theorem 2 establishes that a contingent claim's delta can be

[3] When $m(t) \neq 1$ for all t, the bounds developed in Lemmas 1, 2, 3 and 4 continue to apply with $T-t$ replaced by $\int_t^T m(\tau)d\tau$ where appropriate.

expressed as the expectation of its delta at maturity under a particular diffusion process for the underlying asset.[4]

Theorem 2. (Bergman (1983)) *Suppose the risk-neutral process for the underlying asset is given by* (7)*. Consider a European contingent claim on this asset whose time* T *contractual payoff is* $g(\cdot)$*. The delta of this claim is given by*

$$V_1(s,t) = E\{g_1(\xi_T^{s,t})\},$$

where the dynamic of ξ_τ *is described by*

$$d\xi_\tau = (r\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau))\, d\tau + z(\xi_\tau,\tau)dB_\tau.$$

Theorem 2 is proved in Appendix A. The stochastic process that determines a claim's delta, *the delta process*, is neither the true process, nor the risk-neutral process.

Definition 1. **(The Delta Process)** *When the risk-neutral process is given by* (7)*, the following process will be said to be the corresponding delta process:*

$$d\xi_\tau = (r(\tau)\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau))\, d\tau + z(\xi_\tau,\tau)dB_\tau. \tag{8}$$

The diffusion parameter of the delta process is common to the true process, the risk-neutral process, and the delta process. The drift of the delta process is the sum of the risk-neutral drift plus a term that depends on that common diffusion parameter, and will exceed (be less than) the risk-neutral drift when $z_1(s,t)$ is positive (negative). In the case of a call option, Theorem 2 specializes to an equality between a call's delta and its probability of finishing in-the-money under the delta process. For a call option, $c(s,T) = g(s) = \max[0, s-k]$, and

$$g_1(s) = \begin{cases} 1, & \text{if } s > k; \\ 0, & \text{if } s < k. \end{cases}$$

Hence, for a call option, Theorem 2 specializes to:

$$c_1(s,t) = E\{g_1(\xi_T^{s,t})\} = \Pr\left(\xi_T^{s,t} > k\right),$$

[4] An extension of Theorem 2 to the case of a multi-dimensional diffusion setting (i.e., a setting with stochastic volatility) can be found in the appendix to Bergman, Grundy and Wiener (1996). An independent derivation of Theorem 2 in a deterministic volatility setting can be found in Carr (1993).

with the dynamic of ξ_τ given by the delta process in (8). Whenever a European-style contingent claim is equivalent to a portfolio of call options, bounding the probability that the delta process finishes above any given level allows one to place a bound on the delta of that contingent claim.

2.1. Theorems 1 and 2 and option deltas

When the F transform is applied to the delta process, the drift of the resultant F process (given in general in (6)) specializes to:[5]

$$\begin{aligned}\psi(F(\xi,t),t) &= a(\tau) \quad \left(\frac{\mu(\xi,\tau)}{z(\xi,\tau)} - \frac{1}{2}z_1(\xi,\tau)\right) \qquad + \qquad F_2(\xi,\tau) \\ &= \ 1 \times \left(\frac{r\xi + z_1(\xi,\tau)z(\xi,\tau)}{z(\xi,\tau)} - \frac{1}{2}z_1(\xi,\tau)\right) - \frac{i}{\sigma(\xi,\tau)} \\ &= \frac{r-i}{\sigma(\xi,\tau)} + \frac{1}{2}z_1(\xi,\tau). \end{aligned} \tag{9}$$

Substitution for F, ψ and a in (3) then gives[6]

$$c_1(s,t) = \Pr\left(\frac{\int_k^{se^{i(T-t)}} \frac{1}{\sigma(xe^{-i(T-t)},t)x}dx + \int_t^T \left(\frac{r-i}{\sigma(\xi_\tau,\tau)} + \frac{1}{2}z_1(\xi_\tau,\tau)\right)d\tau}{\sqrt{T-t}} > \mathcal{X}\right), \tag{10}$$

where ξ_τ starts at s and time t and follows the delta process in (8), and $\mathcal{X}$ is determined by the Wiener component of (8). With the aid of a change of variables, $q := xe^{-T}$, the call's delta can be expressed entirely in terms of the real value of the underlying asset, se^{-it}, the real value of the exercise price, ke^{-iT}, the function, $v(\cdot)$, that relates the volatility to the real value of the underlying asset, and the real risk-free rate, $r-i$; i.e., entirely in real terms.

$$c_1(s,t) = \Pr\left(\frac{\int_{ke^{-iT}}^{se^{-it}} \frac{1}{v(q)q}dq + \int_t^T \left(\frac{r-i}{v(\xi_\tau e^{-i\tau})} + \frac{1}{2}\left(v_1(\xi_\tau e^{-i\tau})\xi_\tau e^{-i\tau} + v(\xi_\tau e^{-i\tau})\right)\right)d\tau}{\sqrt{T-t}} > \mathcal{X}\right).$$

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[5] In a Black-Scholes setting with a constant volatility, $\hat{\sigma}$, $z_1(s,t) = \hat{\sigma}$ and expression (9) simplifies to $\psi(F(\xi,t),t) = \frac{r-i+\frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}$.

[6] In a Black-Scholes setting with a constant volatility, $\hat{\sigma}$, evaluation of the integrals in (10) yields the familiar Black-Scholes expression for the hedge ratio:

$$c_1(s,t) = N\left(\frac{\ln(s/k) + \left(r + \frac{1}{2}\hat{\sigma}^2\right)(T-t)}{\hat{\sigma}\sqrt{T-t}}\right).$$

What is important is that when $i \neq 0$, expression (10) can not be re-expressed purely in terms of nominal prices and nominal interest rates alone unless the volatility function $\sigma(\cdot)$ does *not* depend on the price level; i.e., it is only in a Black-Scholes type setting that option prices can be written in terms of nominal prices and nominal interest rates without reference to the inflation rate.

2.2. Bounds on deltas given only limited knowledge of the volatility parameter

Suppose that at some time t we know the underlying asset's current volatility, $\sigma(s,t)$, and/or $\sigma(ke^{-i(T-t)},t)$. Suppose also that we have the additional knowledge that the volatility parameter satisfies one or more of the following restrictions for all s and t:

(*i*) $$\frac{\sigma_1(s,t)s}{\sigma(s,t)} \geq -1.$$

(*ii*) $$\sigma_1(s,t) \geq 0.$$

(*iii*) $$\sigma_1(s,t) \leq 0.$$

(*iv*) $$\underline{\sigma} \leq \sigma(s,t) \leq \overline{\sigma}.$$

Clearly (*iv*) is always trivially satisfied with $\underline{\sigma} = 0$ and $\overline{\sigma}$ infinite. Note that satisfaction of restriction (*ii*) immediately implies that restriction (*i*) is satisfied. Restriction (*i*) implies that the diffusion parameter is non-decreasing in the asset price; i.e., $z_1(s,t) \geq 0$ for all s and t. Under restriction (*i*), σ_1 can be negative, but not 'too' negative; e.g., restriction (*i*) is satisfied in the constant elasticity of variance (CEV) setting studied in Cox (1975). In this CEV setting the diffusion parameter takes the form $z(s,t) := \breve{\sigma}(t)s^\rho$, where $0 \leq \rho \leq 1$, $\breve{\sigma}(t)$ is the time t volatility given a nominal stock price of unity, and the elasticity of volatility, $\omega(s,t)$, is $\omega(s,t) = \rho - 1$. Since $\rho \geq 0$, $\omega(s,t) \geq -1$. Finally, note that since $z(0,t) = 0$ for all limited liability assets, a risky, limited liability asset could never satisfy a restriction that $z_1(s,t) \leq 0$ for all s and t.

How can we use our limited knowledge of the volatility parameter to bound the value for delta given by (10)? As one illustration, suppose we know that the volatility parameter satisfies restriction (*i*). When $z_1(s,t) \geq 0$ for all s and t *and* $r \geq i$, the drift of the F process given in (9) is non-negative. A deterministic lower bound on the drift then takes the simple form $l(t) := 0$; i.e., when ξ_τ follows the delta process, the distribution of $F(\xi_T^{s,t}, T) - F(s,t)$ first order stochastically dominates the distribution of increments over an interval of length $T-t$ in a zero drift, unit diffusion parameter, arithmetic Brownian motion process. Knowledge that $z_1(s,t) \geq 0$ for all s and t also allows us to obtain a lower bound on $F(s,t) - F(k,T)$.

$$\begin{aligned}
L(s,t) &:= \frac{s - ke^{-i(T-t)}}{\sigma(s,t)s} \\
&= \int_k^{se^{i(T-t)}} \frac{1}{\sigma(s,t)se^{i(T-t)}} dx \\
&\leq \int_k^{se^{i(T-t)}} \frac{1}{\sigma(xe^{-i(T-t)},t)x} dx \\
&= F(s,t) - F(k,T).
\end{aligned}$$

Substituting for $l(t)$ and $L(s,t)$ in (4) immediately provides a lower bound on an option's delta that is satisfied whenever $r \geq i$ and $z_1(s,t) \geq 0$ for all s and t. This bound is the special case of the first of the bounds set out in Lemma 1 below, applicable when $\overline{\sigma}$ is infinite and $r \geq i$.

Lemma 1. (Bounds on Deltas) *Suppose that the risk-neutral process for the underlying asset starts at s at time t and follows the diffusion given in (7) with $\underline{\sigma} \leq \sigma(s,t) \leq \overline{\sigma}$ for all s and t. Suppose also that the restrictions of the Assumption Set are satisfied. Let $M := \max\left[\frac{r-i}{\underline{\sigma}}, \frac{r-i}{\overline{\sigma}}\right]$ and $m = \min\left[\frac{r-i}{\underline{\sigma}}, \frac{r-i}{\overline{\sigma}}\right]$.*

I. *Suppose that $z_1(s,t) \geq 0$ for all s and t. A call's delta is then bounded below as:*

$$c_1(s,t) \geq N\left(\frac{\frac{s-ke^{-i(T-t)}}{s} + m\,\sigma(s,t)(T-t)}{\sigma(s,t)\sqrt{T-t}}\right). \tag{11}$$

II. *Suppose that $\sigma_1(s,t) \geq 0$ for all s and t. A call's delta is then bounded below as:*

$$c_1(s,t) \geq N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(m + \frac{1}{2}\underline{\sigma}\right)\sigma(s,t)(T-t)}{\sigma(s,t)\sqrt{T-t}}\right). \tag{12}$$

III. *Suppose that $\sigma_1(s,t) \leq 0$ for all s and t. A call's delta is then bounded above as:*

$$c_1(s,t) \leq N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(M + \frac{1}{2}\overline{\sigma}\right)\sigma(ke^{-i(T-t)},t)(T-t)}{\sigma(ke^{-i(T-t)},t)\sqrt{T-t}}\right). \tag{13}$$

Lemma 1 is proved in Appendix A.

The $\min[\cdot,\cdot]$ and $\max[\cdot,\cdot]$ functions in (11), (12) and (13) reflect the fact that, in general, the real risk-free rate, $r-i$, could be either positive or negative.[7] It is important to recognize how $\sigma(s,t)$ enters the right-hand-side on inequalities (11) and (12). Obviously, if one knew $\sigma(s,t)$ for all s and t, then one could determine option deltas and prices exactly and (11) and (12) would be simply quick consistency checks on one's numerical solution technique. But, when one knows only the volatility at the *current* price and time and that $z_1(s,t) \geq 0$ or $\sigma_1(s,t) \geq 0$ for all s and t, then (11) and (12) become the focus. What if one does not know the volatility at the current price and time? Then one uses the Lemma 1 precondition, that $\underline{\sigma} \leq \sigma(s,t) \leq \overline{\sigma}$ for all s and t, to substitute the appropriate bound on volatility for $\sigma(s,t)$ in the right-hand-side of (11) and (12). Even when the known bounds on volatilty are simply $0 \leq \sigma(s,t) \leq \infty$, expressions (11) and (12) are not without content. For example, as shown in detail in the following subsection, whenever the real risk-free rate is non-negative and $z_1(s,t) \geq 0$ for all s and t, then both (11) and (12) state that the delta of a call is at least 1/2 provided the real value of the underlying exceeds the real value of the option's exercise price.

Tables IA and IB demonstrate that the bounds in Lemma 1 can be quite tight in interesting settings. Table IA compares the true deltas of call options in a CEV world to the lower bound given in (11).[8] The bound in (11) is relevant since in a CEV world, $z_1(s,t) \geq 0$ for all s and t. The risk-neutral CEV process for the asset's nominal price is assumed to take the form

$$d\xi_\tau = r\xi_\tau d\tau + \check{\sigma}\xi_\tau^\rho dB_\tau$$

which, given the Section 1.1. discussion of the time-dependence of volatility, implies that the inflation rate must be zero. Since $\overline{\sigma}$ is infinite in a CEV setting, a positive nominal risk-free rate

[7] Note that a Black-Scholes setting simultaneously satisfies both the restriction that $\sigma_1(s,t) \geq 0$ for all s and t and that $\sigma_1(s,t) \leq 0$ for all s and t, with $\hat{\sigma} = \underline{\sigma} = \sigma(s,t) = \sigma(ke^{-i(T-t)},t) = \overline{\sigma}$. Substituting $\hat{\sigma}$ into the right-hand-sides of inequalities (12) and (13) gives

$$c_1(s,t) = N\left(\frac{\ln(s/k) + \left(r + \frac{1}{2}\hat{\sigma}^2\right)(T-t)}{\hat{\sigma}\sqrt{T-t}}\right).$$

[8] The Mathematica 3.0 program that calculates the true deltas is available upon request.

implies that the min$[\cdot,\cdot]$ term in (11) is zero, and the inequality in (11) simplifies to

$$c_1(s,t) \geq N\left(\frac{s-k}{s\sigma(s,t)\sqrt{T-t}}\right).$$

The first column of Table IA reports this lower bound. The remaining columns report the options' true deltas for various values of the elasticity parameter ρ. As ρ varies, the value of $\check{\sigma}$ is altered so that the *current* volatility is held constant at its assumed known value.

Table IB compares the true deltas of call options in a displaced diffusion world to the lower bound given in (12). The calls are assumed to be written on a non-dividend-paying, unlevered firm that has both a riskless and a risky division. The volatility of the risky division's assets is a constant, $\hat{\sigma}$. The fraction of the firm's assets currently held in the riskless division is f, and no transfers between the two divisions will occur during the life of the option. In this displaced diffusion world, $\sigma_1(s,t) \geq 0$ for all s and t and hence the bound in (12) is relevant. When $i = r$ the bound simplifies to

$$c_1(s,t) \geq N\left(\frac{\ln\left(s/ke^{-r(T-t)}\right)}{\sigma(s,t)\sqrt{T-t}}\right).$$

The first column on Table IB reports this lower bound. The remaining columns report the options' true deltas for various values of f. As f varies, the value of $\hat{\sigma}$ is altered so that the firm's *current* volatility is held constant at its assumed known value.[9]

We now turn to the implications of Lemma 1 for the deltas of in-the-money options.

2.3. Deltas of 'In-the-Money' Options

In the familiar Black-Scholes setting, the delta of any option with $ke^{-r(T-t)} \leq s$ is at least 1/2. In this section we show that this result need not extend beyond a Black-Scholes setting. There are two reasons for this. The first is that when volatility does depend on the price level, then one must be careful to relate the properties of option to real prices and real interest rates. The second

[9] If $r \neq i$, $\overline{\sigma}$ is a determinant of the lower bound in (12). In this displaced diffusion setting $\overline{\sigma} = \hat{\sigma}$. For comparability across columns we adjust the value of $\hat{\sigma}$ so as to hold constant the firm's current volatility. Absent the simplifying assumption that $i = r$, a different lower bound would be applicable for each column.

reason is that even when $i = 0$, if the asset's volatility declines fast enough as its price increases, then the delta of an at-the-money option can be less than 1/2.

We first use the F transform of the delta process to establish that given a one-dimensional diffusion process with $z_1(s,t) \geq 0$ for all s and t and non-negative real interest rates, then the delta of an option is always at least 1/2 provided that the real value of the underlying asset, se^{-it}, is at least equal to the real exercise price, ke^{-iT}; i.e., provided $s \geq ke^{-i(T-t)}$. We then show by example that when $z_1(s,t) \not\geq 0$ for all s and t, then the delta of an at-the-money option can be less than 1/2 even if $r \geq i$.

Definition 2. (At-the-Real-Money) *An option will be said to be at-the-real-money if* $s = ke^{-i(T-t)}$.

Corollary. (Deltas of At-the-Real-Money Calls) *When the risk-neutral process is given by (7),* $r \geq i$ *and* $z_1(s,t) \geq 0$ *for all* s *and* t*, then the delta of an at-the-real-money call option is always at least 1/2.*

Proof: In the definition of F in (2), set $\alpha(t) = e^{i(T-t)}$ and $A(t) = k$. For $s = ke^{-i(T-t)}$, we have $F(s,t) = 0$ and $F(k,T) = 0$. The probability that, starting from s, the delta process finishes above $k = se^{i(T-t)}$ is therefore the same as the probability that the F transform of the delta process finishes above zero when it starts from zero. When $r \geq i$ and $z_1(s,t) \geq 0$ for all s and t, the drift of the F transform of the delta process (given in (9)) is always non-negative. Thus the distribution of $F(\xi_T^{s,t}, T) - F(s,t)$ first-order stochastically dominates the distribution of the increment over the same interval in a mean zero, unit diffusion parameter, Brownian motion process. Hence the probability that F finishes above its starting value is at least 1/2. ∎

It is important to note that the delta of an at-the-real-money option is *not* always at least 1/2, even when $r \geq i$. The restriction that $z_1(s,t) \geq 0$ for all s and t is an important precondition of the above Corollary. When this restriction is not satisfied, the delta of an at-the-real-money option can be less than 1/2. We demonstrate this result by example. Suppose $r = i = 0$, and consider an asset price process that at all times in the interval $[T-0.1, T]$ follows the diffusion

$$ds_\tau = \sigma(s_\tau, \tau) s_\tau dB_\tau,$$

with (14)

$$[\sigma(s,t)]^2 = \frac{e^{2(1-s)}}{1 + s^2(T-t)e^{2(1-s)} + 2s\ln(s) - s(T-t)e^{2(1-s)} + s^2[\ln(s)]^2 - \frac{1}{4}s^2(T-t)^2 e^{4(1-s)}}.$$

We show in Appendix B that this process is well defined for $\tau \in [T-0.1, T]$.[10] Figure 1a depicts the volatility as a function of s for a fixed t. For sufficiently high values of s, the volatility declines faster than $1/s$, and hence the diffusion parameter depicted in Figure 1b is decreasing in s.

Consider a call option on this asset with an exercise price $k = 1$ and a date T maturity. The price of this option is given by the solution of the p.d.e.

$$c_2(s,t) + \frac{1}{2}[\sigma(s,t)]^2 s^2 c_{11}(s,t) = 0,$$

subject to the terminal condition $c(s,T) = \max[0, s-1]$. An analytical solution for the value of this call exists and is given by

$$c(s,t) = sN(\mathrm{d}_1) - N(\mathrm{d}_2),$$

where

$$\mathrm{d}_1 := \frac{\ln(s) + \frac{1}{2}e^{2(1-s)}(T-t)}{e^{1-s}\sqrt{T-t}}$$

and

$$\mathrm{d}_2 := \mathrm{d}_1 - e^{1-s}\sqrt{T-t}.$$

Thus we can determine analytically that when this call option is at-the-real-money, its delta is given by

$$c_1(s,t)\Big|_{s=k=1} = N\left(\frac{\sqrt{T-t}}{2}\right) - \frac{\sqrt{T-t}}{\sqrt{2\pi}} e^{-(T-t)/8}. \quad (15)$$

Figure 2 graphs the right-hand-side of (15) as a function of $T-t$: For *all* $t \in [T-0.1, T)$, the delta of this at-the-money call option is strictly *less* than 1/2.

[10] The volatility parameter in (14) does not take the form $\sigma(s,t) = m(t)v(se^{-it})$.

3. Bounds on State Prices

In this section we show how to use limited information about an asset's volatility parameter to bound state prices in a one-dimensional diffusion world.[11] Let $\pi(s,t,k,T)$ denote the time t price of a state claim written on a non-dividend-paying asset worth s at time t, with that claim paying one dollar at date T if and only if the asset's date T nominal value exceeds k. The price of a state claim is simply the discounted expectation that the underlying asset's nominal price will finish above k under the risk-neutral process; i.e.,

$$\pi(s,t,k,T) = e^{-r(T-t)} \Pr\left(\xi_T^{s,t} > k\right),$$

where ξ follows the diffusion in (7).

Suppose that we have limited information about $\sigma(s,t)$, as in (i), (ii), or (iii). We can apply Theorem 1 to the problem of bounding the prices of state-contingent claims. When the F transform is applied to the risk-neutral process, the drift of the resultant F process (the general form of which is given in (6)) specializes to:

$$\psi(F(\xi,t),t) = \frac{r-i}{\sigma(\xi,\tau)} - \frac{1}{2} z_1(\xi,\tau). \tag{16}$$

Following arguments analogous to those in the proof of Lemma 1, we obtain:[12]

Lemma 2. (Bounds on State Prices) *Suppose that the risk-neutral process for the underlying asset starts at s at time t and follows the diffusion given in* (7) *with $\underline{\sigma} \leq \sigma(s,t) \leq \overline{\sigma}$ for all s and*

[11] Recent theoretical and empirical work on state prices includes Bick and Reisman (1994), Derman and Kani (1994), Dupire (1994), Rubinstein (1994), Aït-Sahalia and Lo (1995), Rady (1995), Jackwerth and Rubinstein (1996), Dumas, Fleming and Whaley (1996), and Jackwerth (1997).

[12] Note that a Black-Scholes setting simultaneously satisfies both the restriction that $\sigma_1(s,t) \geq 0$ for all s and t and that $\sigma_1(s,t) \leq 0$ for all s and t, with $\hat{\sigma} = \underline{\sigma} = \sigma(s,t) = \sigma(ke^{-i(T-t)},t) = \overline{\sigma}$. Substituting the Black-Scholes volatility parameter into the right-hand-sides of inequalities (18) and (19) gives the familiar Black-Scholes result:

$$\pi(s,t,k,T) = e^{-r(T-t)} N\left(\frac{\ln(s/k) + \left(r - \frac{1}{2}\hat{\sigma}^2\right)(T-t)}{\hat{\sigma}\sqrt{T-t}}\right).$$

t. *Suppose also that the restrictions of the Assumption Set are satisfied. Let* $M := \max\left[\frac{r-i}{\underline{\sigma}}, \frac{r-i}{\overline{\sigma}}\right]$ *and* $m := \min\left[\frac{r-i}{\underline{\sigma}}, \frac{r-i}{\overline{\sigma}}\right]$.

I. *Suppose that* $z_1(s,t) \geq 0$ *for all* s *and* t. *State prices are then bounded above as:*

$$\pi(s,t,k,T) \leq e^{-r(T-t)} N\left(\frac{\frac{s-ke^{-i(T-t)}}{ke^{-i(T-t)}} + M\,\sigma(ke^{-i(T-t)},t)(T-t)}{\sigma(ke^{-i(T-t)},t)\sqrt{T-t}}\right). \tag{17}$$

II. *Suppose that* $\sigma_1(s,t) \geq 0$ *for all* s *and* t. *State prices are then bounded above as:*

$$\pi(s,t,k,T) \leq e^{-r(T-t)} N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(M - \frac{1}{2}\underline{\sigma}\right)\sigma(ke^{-i(T-t)},t)(T-t)}{\sigma(ke^{-i(T-t)},t)\sqrt{T-t}}\right). \tag{18}$$

III. *Suppose that* $\sigma_1(s,t) \leq 0$ *for all* s *and* t. *State prices are then bounded below as:*

$$\pi(s,t,k,T) \geq e^{-r(T-t)} N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(m - \frac{1}{2}\overline{\sigma}\right)\sigma(s,t)(T-t)}{\sigma(s,t)\sqrt{T-t}}\right). \tag{19}$$

Corollary. (State Prices) *When the risk-neutral process is given by* (7), $r \geq i$ *and* $z_1(s,t) \geq 0$ *for all* s *and* t, *then the price of a state claim paying one dollar if and only if the underlying finishes above its inflation-adjusted starting value (i.e., above* $se^{i(T-t)}$*) is never greater than the discounted value of fifty cents.*

The observation that the drift of the delta process differs from the drift of the risk-neutral process by $z_1(\xi,t)z(\xi,t)$, suggests the possibility of bounding option deltas in terms of state prices whenever $z_1(s,t) \geq 0$ for all s and t.

3.1. A Relation Between Deltas and State Prices

We introduce the notation $c(s,t,k,T)$ to make explicit the dependence of the option's value on its exercise price.

Theorem 3. *Suppose that the risk-neutral process for the underlying asset starts at* s *at time* t *and follows the diffusion in* (7). *Suppose also that the restrictions of the Assumption Set are satisfied. If* $z_1(s,t) \geq 0$ *for all* s *and* t, *then a call's delta exceeds* $e^{r(T-t)}$ *times the corresponding state price; i.e.*

$$c_1(s,t,k,T) \geq e^{r(T-t)}\pi(s,t,k,T). \tag{20}$$

Proof: Compare the drift of the risk-neutral process in (7) and the drift of the delta process in (8). Given the restriction on the sign of z_1, the drift in (8) exceeds that in (7). The result then follows from Proposition 2.18 of Chapter 5 of Karatzas and Shreve (1991). ▮

Theorem 3 can be applied whenever observed option prices provide a lower bound on $\pi(s,t,k,T)$. Bick and Reisman (1994), Derman and Kani (1994), Dupire (1994), Rubinstein (1994), and Jackwerth and Rubinstein (1996) show that, in the one-dimensional diffusion setting of interest here, a rich enough set of observed option prices could be used to exactly determine an option's delta. But what if one can only observe the prices of two otherwise identical calls with differing exercise prices?[13] Breeden and Litzenberger (1978) show that

$$\pi(s,t,k,T) = -c_3(s,t,k,T). \tag{21}$$

Since a call's price is a convex function of its exercise price, the observed prices of two otherwise identical calls with exercise prices k' and $k'' > k'$ yield

$$-c_3(s,t,k',T) \geq \frac{c(s,t,k',T) - c(s,t,k'',T)}{k'' - k'}. \tag{22}$$

This bound on the state price is depicted in Figure 3. Combining (20), (21) and (22) we can translate the bound on the relevant state price into an empirical bound on the delta of the lower exercise price call whenever $z_1(s,t) \geq 0$ for all s and t:

$$c_1(s,t,k',T) \geq e^{r(T-t)} \frac{c(s,t,k',T) - c(s,t,k'',T)}{k'' - k'}.$$

4. Bounds on VaR

A natural application of Theorem 1 is to the analysis of the true likelihood that a portfolio's value will exceed some critical level on a future date T. To illustrate this application suppose that the true price process for the underlying portfolio is a diffusion of the form

$$d\xi_\tau = \theta(\cdot)\xi_\tau d\tau + \sigma(\xi_\tau,\tau)\xi_\tau dB_\tau, \tag{23}$$

[13] One immediate bound on the delta of a call given only its observed price is provided by Proposition 2 of Bergman, Grundy and Wiener (1996). Proposition 2 establishes that when the underlying asset follows a one-dimensional diffusion (as in the setting considered in this article), then $c(s,t)/s \leq c_1(s,t) \leq (c(s,t) + e^{-r(T-t)}K)/s$ for all s and t.

and an investor is confident that $\underline{\theta} \le \theta(\cdot) \le \overline{\theta}$. The investor plans to hold the portfolio from t to T, and is interested in determining the size of the possible losses that have, say, at least a 1-in-20 chance of occurrence. In addition to the true price process, it is useful to consider the following two diffusion processes for ξ' and ξ'', defined respectively as

$$d\xi'_\tau = \underline{\theta}\xi'_\tau d\tau + \sigma(\xi'_\tau, \tau)\xi'_\tau dB_\tau, \tag{24a}$$

and

$$d\xi''_\tau = \overline{\theta}\xi''_\tau d\tau + \sigma(\xi''_\tau, \tau)\xi''_\tau dB_\tau. \tag{24b}$$

From Proposition 2.18 of Chapter 5 of Karatzas and Shreve (1991), ${\xi''_T}^{s,t}$ first-order stochastically dominates $\xi_T^{s,t}$, which in turn first-order stochastically dominates ${\xi'_T}^{s,t}$. Using (24) and arguments analogous to those in the proof of Lemma 2, limited information about the volatility parameter can be used to obtain bounds on the probabilities of events under the true price process. These bounds are set out in Lemma 3.

Lemma 3. (Bounds on True Probabilities) *Suppose that the true process for the underlying asset starts at s at time t and follows the diffusion given in (23), with $\underline{\theta} \le \theta(\cdot) \le \overline{\theta}$ and $\underline{\sigma} \le \sigma(s,t) \le \overline{\sigma}$ for all s and t. Suppose also that the restrictions of the Assumption Set are satisfied. Let $M := \max\left[\frac{\overline{\theta}-i}{\underline{\sigma}}, \frac{\overline{\theta}-i}{\overline{\sigma}}\right]$ and $m := \min\left[\frac{\underline{\theta}-i}{\underline{\sigma}}, \frac{\underline{\theta}-i}{\overline{\sigma}}\right]$.*

I. *Suppose that $z_1(s,t) \ge 0$ for all s and t.*

$$\Pr(\xi_T^{s,t} > k) \le N\left(\frac{\frac{s-ke^{-i(T-t)}}{ke^{-i(T-t)}} + M\,\sigma(ke^{-i(T-t)},t)(T-t)}{\sigma(ke^{-i(T-t)},t)\sqrt{T-t}}\right).$$

II. *Suppose that $\sigma_1(s,t) \ge 0$ for all s and t.*

$$\Pr(\xi_T^{s,t} > k) \le N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(M - \frac{1}{2}\underline{\sigma}\right)\sigma(ke^{-i(T-t)},t)(T-t)}{\sigma(ke^{-i(T-t)},t)\sqrt{T-t}}\right).$$

III. *Suppose that $\sigma_1(s,t) \le 0$ for all s and t.*

$$\Pr(\xi_T^{s,t} > k) \ge N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(m - \frac{1}{2}\overline{\sigma}\right)\sigma(s,t)(T-t)}{\sigma(s,t)\sqrt{T-t}}\right).$$

Each of Lemmas 1, 2 and 3 presume knowledge of the current volatility, $\sigma(s,t)$, and/or what the current volatility would be if the asset's current real value were equal to the real value of the future critical level k; i.e., $\sigma(ke^{-i(T-t)},t)$. Weaker bounds apply absent this knowledge. As an illustration of these weaker bounds, continue to consider the investor's problem of bounding the VaR associated with investing in the portfolio whose dynamic is given by (23).

Definition 3. (Value-at-Risk) *$VaR(\kappa)$ is the negative of the κ% quantile of the portfolio's return distribution; i.e., the size of the percentage loss with the property that losses of that size or greater have a 1-in-$\frac{100}{\kappa}$ chance of occurrence.*

Bounds on $VaR(\kappa)$ are set out in Lemma 4 and the proof contained in Appendix A.

Lemma 4. (Bounds on VaR) *Suppose that the true process for the underlying asset starts at s at time t and follows the diffusion given in (23), with $\underline{\theta} \leq \theta(\cdot) \leq \overline{\theta}$ and $\underline{\sigma} \leq \sigma(s,t) \leq \overline{\sigma}$ for all s and t. Suppose also that the restrictions of the Assumption Set are satisfied. Let $M := \max\left[\frac{\overline{\theta}-i}{\underline{\sigma}}, \frac{\overline{\theta}-i}{\overline{\sigma}}\right]$ and $m := \min\left[\frac{\underline{\theta}-i}{\underline{\sigma}}, \frac{\underline{\theta}-i}{\overline{\sigma}}\right]$. Let $\Phi(\kappa)$ denote the $(1-\kappa)$% quantile of the standard normal density function.*

I. *Suppose that $z_1(s,t) \geq 0$ for all s and t.*

$$VaR(\kappa) \geq \begin{cases} 1 - exp\left((i + M\underline{\sigma})(T-t) - \Phi(\kappa)\underline{\sigma}\sqrt{T-t}\right), & if \ \ M\sqrt{T-t} \leq \Phi(\kappa); \\ 1 - exp\left((i + M\overline{\sigma})(T-t) - \Phi(\kappa)\overline{\sigma}\sqrt{T-t}\right), & if \ \ M\sqrt{T-t} > \Phi(\kappa). \end{cases} \tag{25}$$

II. *Suppose that $\sigma_1(s,t) \geq 0$ for all s and t.*

$$VaR(\kappa) \geq \begin{cases} 1 - exp\left(\left(i + \left(M - \frac{1}{2}\underline{\sigma}\right)\underline{\sigma}\right)(T-t) - \Phi(\kappa)\underline{\sigma}\sqrt{T-t}\right), & if \ \ \left(M - \frac{1}{2}\underline{\sigma}\right)\sqrt{T-t} \leq \Phi(\kappa); \\ 1 - exp\left(\left(i + \left(M - \frac{1}{2}\underline{\sigma}\right)\overline{\sigma}\right)(T-t) - \Phi(\kappa)\overline{\sigma}\sqrt{T-t}\right), & if \ \ \left(M - \frac{1}{2}\underline{\sigma}\right)\sqrt{T-t} > \Phi(\kappa). \end{cases} \tag{26}$$

III. *Suppose that $\sigma_1(s,t) \leq 0$ for all s and t.*

$$VaR(\kappa) \leq \begin{cases} 1 - exp\left(\left(i + \left(m - \frac{1}{2}\overline{\sigma}\right)\overline{\sigma}\right)(T-t) - \Phi(\kappa)\overline{\sigma}\sqrt{T-t}\right), & if \ \ \left(m - \frac{1}{2}\overline{\sigma}\right)\sqrt{T-t} \leq \Phi(\kappa); \\ 1 - exp\left(\left(i + \left(m - \frac{1}{2}\overline{\sigma}\right)\underline{\sigma}\right)(T-t) - \Phi(\kappa)\underline{\sigma}\sqrt{T-t}\right), & if \ \ \left(m - \frac{1}{2}\overline{\sigma}\right)\sqrt{T-t} > \Phi(\kappa). \end{cases} \tag{27}$$

Percentage losses with a 1-in-20 chance of occurrence can be bounded by setting $\Phi(\kappa) = 1.645$ in Lemma 4. Losses with a 1-in-100 chance of occurrence correspond to $\Phi(\kappa) = 2.326$. Table II reports one application of the bounds in Lemma 4. Given some limited information about an asset's price diffusion, an investor is assumed to be interested in determining bounds on the size of the percentage loss with the property that losses of that size or greater have a 1-in-20 chance of occurrence; i.e., Table II reports bounds on $VaR(5)$. An investor believes that $\underline{\theta} = \overline{\theta}$ and that this instantaneous expected rate of return is 10% per annum. The inflation rate is 5% per annum. The investor also believes that the volatility parameter satisfies restrictions (i), (iii) and (iv) for all s and t; i.e., for all s and t, $z_1(s,t) \geq 0$, $\sigma_1(s,t) \leq 0$, and $\sigma(s,t) \in [\underline{\sigma}, \overline{\sigma}]$. The Table reports upper and lower bounds on $VaR(5)$ for three sets of $[\underline{\sigma}, \overline{\sigma}]$ values. Each pair of columns reports the lower bound given in (25) and the upper bound given in (27) for a particular value of $[\underline{\sigma}, \overline{\sigma}]$. The rows correspond to holding periods of 1 trading day, 1 week, 1 month, 1 quarter and 1 year. To read the Table, focus on the pair in the upper-left-hand corner: If the annual volatility is known to be between 10% and 20%, then overnight losses of 0.995% or more will occur at least 5% of the time, while overnight losses of 2.028% or more will occur at most 5% of time. Clearly, tighter bounds on VaR could be obtained if the investor knew more; e.g., if the investor knew the asset's current volatility. The bounds on true probabilities given in Lemma 3 imply bounds on VaR that are tighter than the bounds of Lemma 4.

5. Summary and Extensions

Theorem 1 establishes the existence of a monotonic function that when applied to an initial one-dimensional diffusion process with a level-dependent volatility parameter yields a new diffusion with a deterministic diffusion parameter. Limited knowledge of the relation between the volatility and the level of the original process is then shown to imply a deterministic bound on the drift of the transformed process. Hence the probability of events under the original process can be bounded in terms of the probability of related events under an arithmetic Brownian motion process. This result is used to provide bounds on deltas (Lemma 1), state prices (Lemma 2), true probabilities (Lemma 3), and VaR (Lemma 4) given only limited knowledge of the functional form of an underlying asset's volatility parameter. In deriving these bounds we recognize that an asset's volatility depends on its real price, and not its nominal price. Hence our bounds are in terms of nominal prices, nominal interest rates *and* the inflation rate. Direct application of Lemma 1 allows us to generalize the familiar Black-Scholes result that the delta of an at-the-money option is always at least 1/2 to all settings where the diffusion parameter is non-decreasing in the underlying's value, the real interest rate is non-negative, and the option is at-the-*real*-money. At-the-real-money means that the real value of the underlying asset is equal to the real value of the option's exercise price. We show that the restriction that the diffusion parameter be non-decreasing in the underlying's value can be critical. We provide a zero inflation rate example in which the stock's volatility decreases so quickly with an increase in its price that, in direct contrast to the result in a Black-Scholes setting, the delta of an at-the-money call is always *less* than 1/2. We also show how the bounds on state prices implied by even a coarse grid of observed option prices can be used to bound option deltas whenever the underlying asset's diffusion parameter is non-decreasing in the asset's value.

There are many possible extensions of this line of research. One concerns using the bounds on deltas and state prices to derive bounds on option prices. For example, given that

$$c(s,t,k,T) = \int_0^s c_1(x,t,k,T)dx,$$

result III of Lemma 1 can be used to place an upper bound on call prices whenever $\sigma_1(s,t) \leq 0$ for all s and t. Similarly, given that

$$c(s,t,k,T) = \int_k^\infty \pi(s,t,x,T)dx,$$

result III of Lemma 2 can be used to place a corresponding lower bound on call prices whenever $\sigma_1(s,t) \leq 0$ for all s and t. A second extension involves combining Theorem 7 of Bergman, Grundy and Wiener (1996) (BGW) and Chebyshev's inequality to bound deltas, state prices and VaR. For example, consider bounding state prices. It follows from BGW Theorem 7 that if ξ_τ starts at s at time t and follows the risk-neutral price process in (7) with $\sigma(s,t) \leq \overline{\sigma}$ for all s and t, then

$$\text{Variance}\left\{\xi_T^{s,t}\right\} \leq s^2 e^{2r(T-t)}\left(e^{\overline{\sigma}^2(T-t)} - 1\right).$$

For $s < ke^{-r(T-t)}$, Chebyshev's inequality can then be used to bound state prices as:

$$\pi(s,t,k,T) \leq e^{-r(T-t)} \frac{e^{\overline{\sigma}^2(T-t)} - 1}{\left(\frac{ke^{-r(T-t)}}{s} - 1\right)^2}.$$

The bounds on deltas developed in Lemma 1 apply to European-style calls. Bounds on the values and deltas of American-style options can also be developed. For example, let $C(s,t,k,T:\delta,T')$ denote the value of an American-style call option written on a stock that will pay a proportional dividend of δ of the stock's value at time $T' < T$. Immediately prior to the ex-dividend date, at time T'^- we have

$$C(s,T'^-,k,T:\delta,T') = \max\left[s-k, c\left(s(1-\delta),T',k,T\right)\right].$$

Suppose that limited information about the functional form of the volatility parameter can be used to place lower bounds on state prices and let Π denote the lower bound on π. These lower bounds on state prices can be used to place a lower bound on $c(s(1-\delta),T',k,T)$. (Alternately, if $\underline{\sigma} \leq \sigma(s,t) \leq \overline{\sigma}$ for all s and t, then after allowing for the time T' dividend, BGW Theorem 8 can be used to bound $c\left(s(1-\delta),T',k,T\right)$.) One can use the lower bound on $c\left(s(1-\delta),T',k,T\right)$ to place a lower bound on $C(s,T'^-,k,T:\delta,T')$. Let $\underline{\mathcal{C}}(s,T'^-,k,T:\delta,T')$ denote this lower bound. Thus we have

$$\begin{aligned} C(s,t,k,T:\delta,T') &\geq \int_0^\infty \frac{\partial\underline{\mathcal{C}}(x,T'^-,k,T:\delta,T')}{\partial x} \times \pi(s,t,x,T')\,dx \\ &\geq \int_0^\infty \frac{\partial\underline{\mathcal{C}}(x,T'^-,k,T:\delta,T')}{\partial x} \times \Pi(s,t,x,T')\,dx\,. \end{aligned}$$

Finally, El Karoui, Jeanblanc-Picque and Shreve (1995) and Bergman (1998) establish that, given an underlying diffusion, American call prices are convex in the value of the underlying asset, and hence

$$C_1(s,t,k,T:\delta,T') \geq \frac{C(s,t,k,T:\delta,T')}{s} \geq \frac{\int_0^\infty \frac{\partial \mathcal{C}(x,T'^-,k,T:\delta,T')}{\partial x} \times \Pi(s,t,x,T')\,dx}{s}\,.$$

Another natural extension of this line of research concerns the issue of bounding option deltas and state prices when one has only limited information about a stock's *stochastic* volatility. Bakshi, Cao and Chen (1997) demonstrate that hedging errors based on an empirically determined stochastic volatility model can be only one half those of, say, a Black-Scholes-based hedge. When a stock's volatility is stochastic, it is common to model the price process as a two-dimensional diffusion. Transformations of a two-dimensional process that yield a second two dimensional process with deterministic diffusion parameters and analytically tractable drift parameters are the subject of our ongoing research.

Appendix A: Observations, Theorems, and Lemmas

Observation 2

The partial derivatives of F are given by

$$F_1(\xi,t) = \frac{a(t)}{z(\xi,t)},$$
$$F_{11}(\xi,t) = -\frac{a(t)z_1(\xi,t)}{[z(\xi,t)]^2},$$

and

$$F_2(\xi,t) = \frac{\alpha_1(t)a(t)}{\sigma(\xi,t)\alpha(t)} - \frac{A_1(t)a(t)}{\sigma(A(t)/\alpha(t),t)A(t)}$$
$$+\int_{A(t)}^{\xi\alpha(t)} \left(\frac{a_1(t)\sigma(x/\alpha(t),t) + a(t)\left(\sigma_1(x/\alpha(t),t)\frac{x}{[\alpha(t)]^2}\alpha_1(t) - \sigma_2(x/\alpha(t),t)\right)}{[\sigma(x/\alpha(t),t)]^2 x} \right) dx.$$

Applying Ito's Lemma to F we have

$$dF(\xi_\tau,\tau) = F_1(\xi_\tau,\tau)d\xi_\tau + \frac{1}{2}F_{11}(\xi_\tau,\tau)[d\xi_\tau]^2 + F_2(\xi_\tau,\tau)d\tau$$
$$= \left(a(\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} - \frac{1}{2}z_1(\xi_\tau,\tau)\right) + F_2(\xi_\tau,t)\right)d\tau + a(\tau)dB_\tau. \qquad (A1)$$

Since $F(\xi,t)$ is strictly increasing in ξ for all t, there exists an inverse function γ such that $\xi = \gamma(F(\xi,t),t)$ for all ξ and t. Thus the drift parameter in $(A1)$ can be expressed as $\psi(F_\tau,\tau)$. The diffusion parameter of the F_τ process is the deterministic value $a(\tau)$.

Observation 2′

Applying Ito's Lemma to $\mathcal{F}$ we have

$$d\mathcal{F}(\xi_\tau,\tau) = \left(a(\tau)\mathcal{F}(\xi_\tau,\tau)\left(\frac{\mu(\xi_\tau,\tau)}{z(\xi_\tau,\tau)} + \frac{1}{2}\left(a(\tau) - z_1(\xi_\tau,\tau)\right)\right) + \mathcal{F}_2(\xi_\tau,t)\right)d\tau + a(\tau)\mathcal{F}(\xi_\tau,\tau)dB_\tau.$$

Proof of Theorem 1

$$\Pr\left(\xi_T^{y,t} > k\right) = \Pr\left(F\left(\xi_T^{y,t},T\right) > F(k,T)\right) = \Pr\left(F(y,t) + \int_t^T dF_\tau > F(k,T)\right)$$
$$= \Pr\left(F(y,t) + \int_t^T \psi(F(\xi_\tau,\tau),\tau)d\tau + \int_t^T a(\tau)dB_\tau > F(k,T)\right)$$
$$= \Pr\left(\frac{F(y,t) - F(k,T) + \int_t^T \psi(F(\xi_\tau,\tau),\tau)d\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}} > \mathcal{X}\right),$$

where the random variable $\mathcal{X} := -\frac{\int_t^T a(\tau) dB_\tau}{\sqrt{\int_t^T [a(\tau)]^2 d\tau}}$ is distributed $\mathcal{N}(0,1)$. ∎

Note that the random variables compared in the inequality are not independent.

Proof of Theorem 2

Assume that g is differentiable on its domain. Under mild regularity conditions the claim's price, $V(s,t)$, solves the p.d.e.:

$$rV_1(s,t)s - rV(s,t) + V_2(s,t) + \frac{1}{2}[z(s,t)]^2 V_{11}(s,t) = 0 \tag{A2}$$

subject to the terminal condition $V(s,T) = g(s)$. Taking the partial of $(A2)$ with respect to s gives

$$rV_{11}(s,t)s + V_{12}(s,t) + z_1(s,t)z(s,t)V_{11}(s,t) + \frac{1}{2}[z(s,t)]^2 V_{111}(s,t) = 0. \tag{A3}$$

Let f be the first partial of the contingent claim's value with respect to the value of the underlying; $f(s,t) := V_1(s,t)$. The p.d.e. in $(A3)$ can then be rewritten as

$$(rs + z_1(s,t)z(s,t))\, f_1(s,t) + f_2(s,t) + \frac{1}{2}[z(s,t)]^2 f_{11}(s,t) = 0.$$

Assuming that $rs + z_1(s,t)z(s,t)$ and $z(s,t)$ each satisfy Lipschitz and growth conditions, the Feynman-Kac Theorem can be used to express $V_1(s,t)$ as

$$V_1(s,t) = E\{g_1(\xi_T^{s,t})\},$$

where the dynamic of ξ_τ is described by

$$d\xi_\tau = (r\xi_\tau + z_1(\xi_\tau,\tau)z(\xi_\tau,\tau))\, d\tau + z(\xi_\tau,\tau) dB_\tau.$$

The generalization to the case where the payoff function g has a left and a right derivative everywhere on its domain, where the two need not be equal, and where one of them may be plus infinity or minus infinity, follows the *Generalization of Theorem 1* in the Appendix to Bergman, Grundy and Wiener (1996).

Proof of Lemma 1 (Bounds on Deltas)

I. We first consider the case where $z_1(s,t) \geq 0$ for all s and t. The drift of the F process obtained when the F transformation in (2) is applied to the delta process is given in (9) as

$$\psi(F(\xi,t),t) = \frac{r-i}{\sigma(\xi_\tau,\tau)} + \frac{1}{2} z_1(\xi_\tau,\tau).$$

$\psi(F(\xi,t),t)$ is bounded below for all ξ and t by m defined in Lemma 1. As shown in section 2.2., $F(s,t) - F(k,T)$ is bounded below by

$$L(s,t) := \frac{s - ke^{-i(T-t)}}{\sigma(s,t)s}.$$

Substituting for $a(t)$, and $L(s,t)$ in (4) gives result I of Lemma 1:

$$\begin{aligned} c_1(s,t) &\geq N\left(\frac{\frac{s-ke^{-i(T-t)}}{\sigma(s,t)s} + \int_t^T m d\tau}{\sqrt{T-t}}\right) \\ &= N\left(\frac{\frac{s-ke^{-i(T-t)}}{s} + m\sigma(s,t)(T-t)}{\sigma(s,t)\sqrt{T-t}}\right). \end{aligned}$$

II. Now consider the case where $\sigma_1(s,t) \geq 0$ for all s and t. Rewrite the drift of the F process as

$$\psi(F(\xi,t),t) = \frac{r-i}{\sigma(\xi_\tau,\tau)} + \frac{1}{2}\left(\sigma_1(\xi_\tau,\tau)\xi_\tau + \sigma(\xi_\tau,\tau)\right).$$

$\psi(F(\xi,t),t)$, is bounded below for all ξ and t by $m + \frac{1}{2}\underline{\sigma}$. Also $F(s,t) - F(k,T)$ is bounded below by

$$\begin{aligned} L(s,t) &:= \frac{\ln\left(s/ke^{-i(T-t)}\right)}{\sigma(s,t)} \\ &= \int_k^{se^{i(T-t)}} \frac{1}{\sigma(s,t)x} dx \\ &\leq \int_k^{se^{i(T-t)}} \frac{1}{\sigma(xe^{-i(T-t)},t)x} dx \\ &= F(s,t) - F(k,T). \end{aligned}$$

Substituting for $a(t)$, $l(t)$ and $L(s,t)$ in (4) gives result II of Lemma 1:

$$\begin{aligned} c_1(s,t) &\geq N\left(\frac{\frac{\ln\left(s/ke^{-i(T-t)}\right)}{\sigma(s,t)} + \int_t^T \left(m + \frac{1}{2}\underline{\sigma}\right) d\tau}{\sqrt{T-t}}\right) \\ &= N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(m + \frac{1}{2}\underline{\sigma}\right)\sigma(s,t)(T-t)}{\sigma(s,t)\sqrt{T-t}}\right). \end{aligned}$$

III. Finally consider the case where $\sigma_1(s,t) \leq 0$ for all s and t. $\psi(F(\xi,t),t)$, is bounded above for all ξ and t by $M + \frac{1}{2}\overline{\sigma}$ (M defined in Lemma 1). $F(s,t) - F(k,T)$ is bounded above by

$$\begin{aligned} U(s,t) &:= \frac{\ln\left(s/ke^{-i(T-t)}\right)}{\sigma(ke^{-i(T-t)},t)} \\ &= \int_k^{se^{i(T-t)}} \frac{1}{\sigma(ke^{-i(T-t)},t)x}dx \\ &\geq \int_k^{se^{i(T-t)}} \frac{1}{\sigma(xe^{-i(T-t)},t)x}dx \\ &= F(s,t) - F(k,T). \end{aligned}$$

Substituting for $a(t)$,and $U(s,t)$ in (4) gives result III of Lemma 1:

$$\begin{aligned} c_1(s,t) &\leq N\left(\frac{\frac{\ln\left(s/ke^{-i(T-t)}\right)}{\sigma(ke^{-i(T-t)},t)} + \int_t^T \left(M + \frac{1}{2}\overline{\sigma}\right)d\tau}{\sqrt{T-t}}\right) \\ &= N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + \left(M + \frac{1}{2}\overline{\sigma}\right)\sigma(ke^{-i(T-t)},t)(T-t)}{\sigma(ke^{-i(T-t)},t)\sqrt{T-t}}\right). \quad \blacksquare \end{aligned}$$

Proof of Lemma 4 (Bounds on VaR)

I. We first consider the case where $z_1(s,t) \geq 0$ for all s and t. When ξ_τ follows the diffusion in (23), and ξ''_τ follows the diffusion in $(24b)$,

$$\Pr(\xi_T^{s,t} > k) \leq \Pr(\xi_T''^{\,s,t} > k). \tag{A4}$$

The drift of the F process obtained when the F function in (2) is applied to the process in $(24b)$ is

$$\psi(F(\xi'',t),t) = \frac{\overline{\theta} - i}{\sigma(\xi'',\tau)} - \frac{1}{2} z_1(\xi'',\tau).$$

$\psi(F(\xi'',t),t)$ is bounded above for all ξ and t by M (defined in Lemma 4). Also $F(s,t) - F(k,T)$ is bounded above by

$$U(s,t) := \begin{cases} \frac{\ln\left(s/ke^{-i(T-t)}\right)}{\underline{\sigma}}, & \text{if } s \geq ke^{-i(T-t)}; \\ \frac{\ln\left(s/ke^{-i(T-t)}\right)}{\overline{\sigma}}, & \text{if } s < ke^{-i(T-t)}. \end{cases}$$

Noting $(A4)$ and substituting for $a(t)$, and $U(t)$ in (5) gives

$$\Pr(\xi_T^{s,t} > k) \leq \begin{cases} N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + M\underline{\sigma}(T-t)}{\underline{\sigma}\sqrt{T-t}}\right), & \text{if } s \geq ke^{-i(T-t)}; \\ N\left(\frac{\ln\left(s/ke^{-i(T-t)}\right) + M\overline{\sigma}(T-t)}{\overline{\sigma}\sqrt{T-t}}\right), & \text{if } s < ke^{-i(T-t)}. \end{cases} \tag{A5}$$

Let $\phi(\kappa)$ denote the critical value of k/s such that the right-hand-side of $(A5)$ is equal to $1 - \frac{\kappa}{100}$. From the definition of $VaR(\kappa)$, we have that

$$VaR(\kappa) \geq 1 - \phi(\kappa).$$

Solving for $\phi(\kappa)$ gives

$$\phi(\kappa) = \begin{cases} exp\left(\left(i + M\underline{\sigma}\right)(T-t) - \Phi(\kappa)\underline{\sigma}\sqrt{T-t}\right), & \text{if } M\sqrt{T-t} \leq \Phi(\kappa); \\ exp\left(\left(i + M\overline{\sigma}\right)(T-t) - \Phi(\kappa)\overline{\sigma}\sqrt{T-t}\right), & \text{if } M\sqrt{T-t} > \Phi(\kappa). \end{cases}$$

This establishes result I of Lemma 4. The proof of results II and III is left to the interested reader.

Appendix B: Demonstration that the SDE in (14) satisfies both a Lipschitz and a Growth Condition

Note first that $[\sigma(s,t)]^2$ as defined in (14) is positive and finite for all $s \geq 0$ and all $t \in [T - 0.1, T]$. This follows from the fact that the denominator in (14) is always positive. In fact, the denominator is always greater than $0.02\overline{6}$. To demonstrate this lower bound on the denominator we introduce a new variable $\mathcal{T}$.

$$\mathcal{T} = \mathcal{T}(s,t) := (T-t)e^{2(1-s)}.$$

We denote the denominator by $\mathcal{Q}$ and rewrite it as

$$\mathcal{Q}(s,\mathcal{T}) = 1 + s^2\mathcal{T} + 2s\ln(s) - s\mathcal{T} + s^2[\ln(s)]^2 - \frac{1}{4}s^2\mathcal{T}^2.$$

We can bound the sum of the three terms involving $\mathcal{T}$ as

$$\begin{aligned} s^2\mathcal{T} - s\mathcal{T} - \frac{1}{4}s^2\mathcal{T}^2 &= s\mathcal{T}\left(s - 1 - \frac{1}{4}s\mathcal{T}\right) \\ = s\mathcal{T}\left(s\left(1-\frac{\mathcal{T}}{4}\right)-1\right) &= s\mathcal{T}\frac{4-\mathcal{T}}{4}\left(s - \frac{4}{4-\mathcal{T}}\right) \\ \geq \frac{2\mathcal{T}}{4-\mathcal{T}}\frac{4-\mathcal{T}}{4}\frac{-2}{4-\mathcal{T}} &= -\frac{\mathcal{T}}{4-\mathcal{T}} > -\frac{1}{3}, \end{aligned}$$

since $0 \leq \mathcal{T} \leq 0.1e^2 < 1$ and the function $\frac{-\mathcal{T}}{4-\mathcal{T}}$ is monotonically decreasing in this region. Thus

$$\mathcal{Q}(s,\mathcal{T}) > \frac{2}{3} + 2s\ln(s) + s^2[\ln(s)]^2.$$

Using the fact that $s\ln(s) > -0.4$ for all values of s we have $\mathcal{Q}(s,\mathcal{T}) > 0.02\overline{6}$.

Now note that the diffusion parameter, $z(s,t) = \sigma(s,t)s$, must satisfy Lipschitz and growth conditions. Consider the respective Lipschitz and growth conditions given in conditions (E.2) and (E.3) of Duffie (1992, p. 240). That a growth condition of the form (E.3) is satisfied for all $t \in [T-0.1, T]$ follows immediately from the twin observations that the numerator of the expression for $[\sigma(s,t)]^2$ in (14) is a decreasing function of s, while the denominator is bounded from below. To demonstrate that a Lipschitz condition of the form in (E.2) is satisfied for all $t \in [T-0.1, T]$, we demonstrate that there exists a constant $\mathcal{K}$ such that

$$z_1(s,t) \leq \mathcal{K}.$$

First we write

$$\sigma(s,t)s = \frac{P(s)}{\sqrt{\mathcal{Q}(s,\mathcal{T}(s,t))}}$$

where $P(s) := se^{1-s}$. Then

$$\frac{\partial[\sigma(s,t)s]}{\partial s} = \frac{P_1 \mathcal{Q}^{1/2} - \frac{1}{2} P \mathcal{Q}^{-1/2} \frac{\partial \mathcal{Q}(s,\mathcal{T}(s,t))}{\partial s}}{\mathcal{Q}}. \tag{B1}$$

Recalling that $\mathcal{Q}(s, \mathcal{T})$ is positive and bounded from below, we have that the first term in the expression in $(B1)$,

$$\frac{P_1(s)}{\sqrt{\mathcal{Q}(s,\mathcal{T})}} = \frac{e^{1-s}(1-s)}{\sqrt{\mathcal{Q}(s,\mathcal{T})}},$$

is bounded. The second term in $(B1)$,

$$\frac{P(s)}{2[\mathcal{Q}(s,\mathcal{T})]^{3/2}} \frac{\partial \mathcal{Q}(s,\mathcal{T}(s,t))}{\partial s}$$

can be rewritten as

$$\frac{se^{1-s}}{2[\mathcal{Q}(s,\mathcal{T})]^{3/2}}(4s\mathcal{T} + 2\ln(s) + 2 - \mathcal{T} + 2s[\ln(s)]^2 + 2s\ln(s) - \frac{1}{2}s\mathcal{T}^2 - 2s^2\mathcal{T} + s^2\mathcal{T}^2),$$

This expression is a continuous function that tends to zero when $s \to +\infty$ and is bounded when $s \to 0$. Thus it is bounded from above and below. The difference between the two terms comprising $(B1)$ is then a continuous function that is bounded from above and below, and the Lipschitz condition is satisfied. Thus we have established that the SDE in (14) is well-defined over the relevant time horizon. ∎

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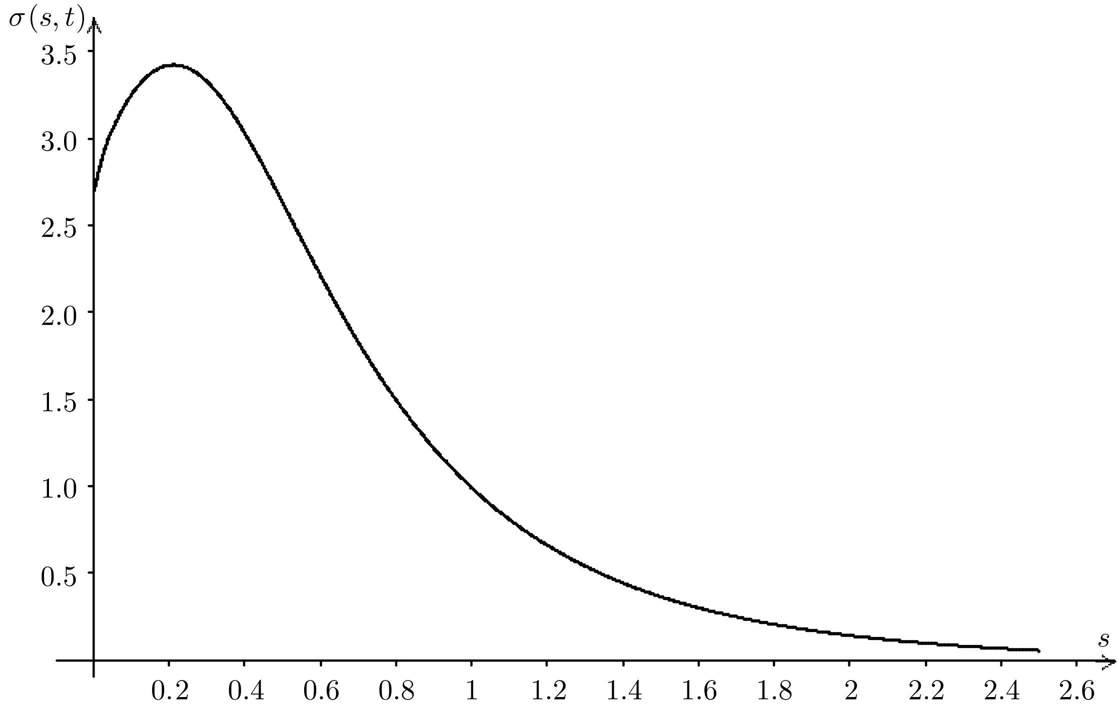

Figure 1a. A volatility function, $\sigma(s,t)$, such that the diffusion parameter, $z(s,t) = \sigma(s,t)\,s$, is not everywhere non-decreasing in s. Volatility depicted at time $t = T - 0.05$, when $\sigma(s,t)$ is defined by

$$[\sigma(s,t)]^2 = \frac{e^{2(1-s)}}{1 + s^2(T-t)e^{2(1-s)} + 2s\ln(s) - s(T-t)e^{2(1-s)} + s^2[\ln(s)]^2 - \frac{1}{4}s^2(T-t)^2e^{4(1-s)}}.$$

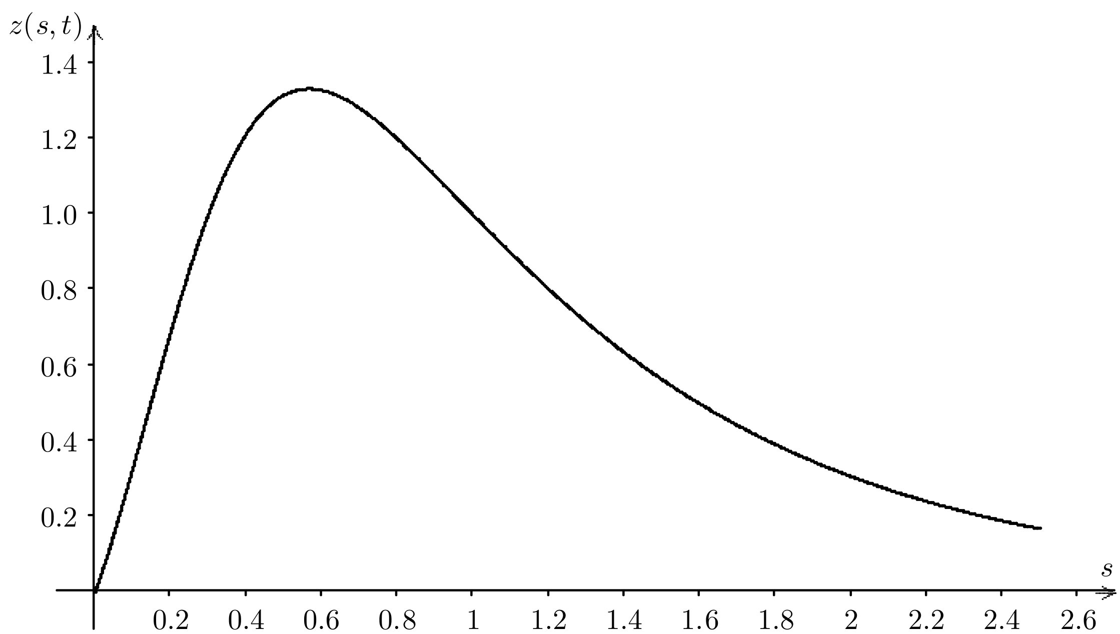

Figure 1b. Diffusion parameter, $z(s,t)$, that is not everywhere non-decreasing in s. $z(s,t) = \sigma(s,t)\,s$, where $\sigma(s,t)$ is as depicted in Figure 1a.

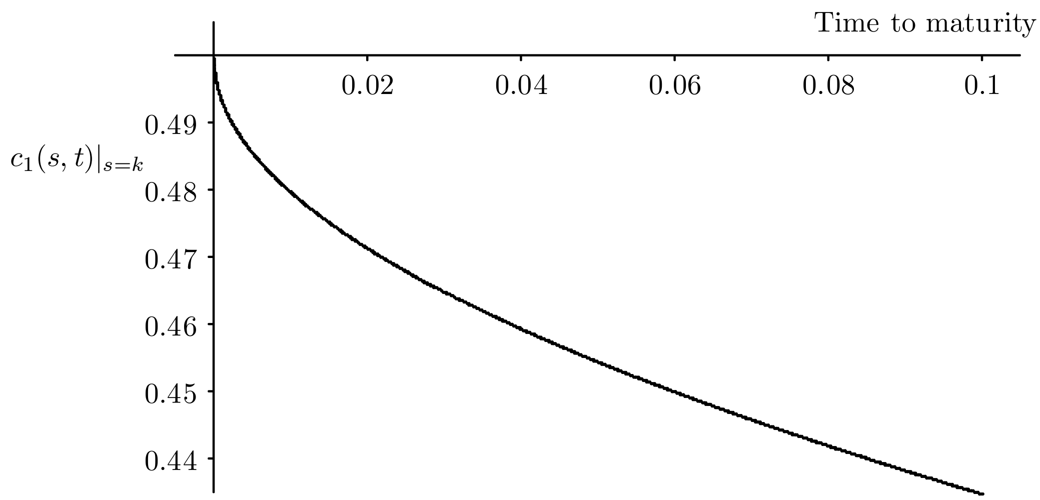

Figure 2. Delta of an at-the-money call with $s = k = 1$. The risk-free rate is zero. The call with maturity date T is written on an asset whose price s_t at all times $t \in [T - 0.1, T]$ follows the risk-neutral diffusion:

$$ds_\tau = \sigma(s_\tau, \tau) s_\tau dB_\tau,$$

whose squared volatility parameter is given by

$$[\sigma(s,t)]^2 = \frac{e^{2(1-s)}}{1 + s^2(T-t)e^{2(1-s)} + 2s\ln(s) - s(T-t)e^{2(1-s)} + s^2[\ln(s)]^2 - \frac{1}{4}s^2(T-t)^2e^{4(1-s)}}.$$

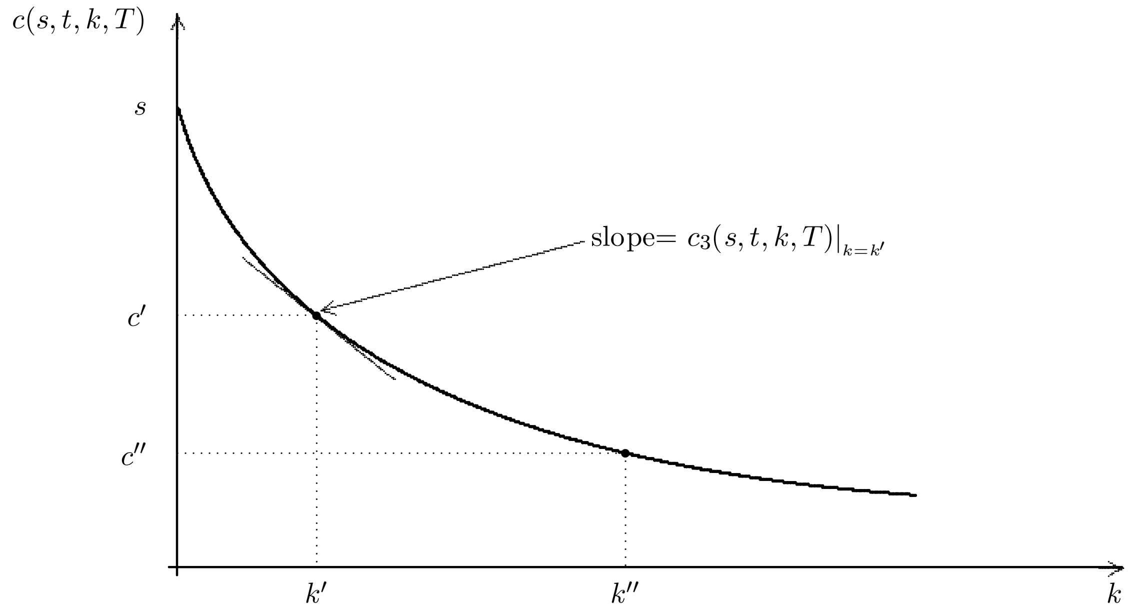

Figure 3. Illustration of the bound on $c_3(s,t,k,T)|_{k=k'}$ implied by (i) the no-arbitrage relation that an option's price must be a convex function of its exercise price and (ii) the observed prices of options with exercise prices of k' and $k'' > k'$. $c(s,t,k,T)$ is the time t price of a call option with a date T maturity and exercise price of k written on an asset worth s.

$$-c_3(s,t,k,T)|_{k=k'} \geq \frac{c(s,t,k',T) - c(s,t,k'',T)}{k'' - k'}.$$

Table I

Comparison of the Lower Bound on a Call's Delta and its True Delta

A. The Underlying Asset's Diffusion Parameter is Non-Decreasing in the Asset's Value.

k	Lower Bound	True Delta				
		$\rho = 0$ (Absolute)	$\rho = 1/4$	$\rho = 1/2$ (Square Root)	$\rho = 3/4$	$\rho = 1$ (Black-Scholes)
\$25	0.993790	0.994566	0.997321	0.998823	0.999552	0.999857
\$35	0.933193	0.939804	0.950381	0.959551	0.967440	0.974165
\$40	0.841345	0.854574	0.866998	0.878570	0.889357	0.899414
\$45	0.691462	0.712048	0.723329	0.734173	0.744614	0.754684
\$50	0.5	0.524918	0.534901	0.544820	0.554695	0.564544
\$55	0.308538	0.332007	0.342746	0.353768	0.365092	0.376741
\$60	0.158655	0.175860	0.187703	0.200135	0.213187	0.226896
\$65	0.066807	0.076627	0.087263	0.098871	0.111501	0.125212
\$75	0.006210	0.007720	0.011474	0.016472	0.022939	0.031103

The risk-neutral process for the nominal price of the underlying asset is the CEV process,

$$ds_\tau = rs_\tau d\tau + \check{\sigma} s_\tau^\rho dB_\tau.$$

Such a process implies that the inflation rate must be zero. As ρ varies, the value of $\check{\sigma}$ is altered so that the current volatility is held constant at 40% per annum; i.e., $\check{\sigma} = 0.4/s^{\rho-1}$. The current stock price is \$50, $r = 0.05$, and the option matures 3 months hence. The option's exercise price is denoted by k. The lower bound follows from result I of Lemma 1.

B. The Underlying Asset's Volatility Parameter is Non-Decreasing in the Asset's Value.

k	Lower Bound	True Delta	
		$f = 0$ (Black-Scholes)	$f = 1/4$
\$25	0.999736	0.999857	0.999993
\$35	0.962737	0.974165	0.983085
\$40	0.867728	0.899414	0.913841
\$45	0.700835	0.754684	0.768837
\$50	0.5	0.564544	0.577681
\$55	0.316841	0.376741	0.391560
\$60	0.180988	0.226896	0.243899
\$65	0.094790	0.125212	0.142239
\$75	0.021314	0.031103	0.042179

The risk-neutral process for the nominal price of the underlying stock is the displaced diffusion process,

$$ds_\tau = rs_\tau d\tau + \hat{\sigma}\left(1 - \frac{Re^{r\tau}}{s_\tau}\right) s_\tau dB_\tau.$$

The fraction of the underlying stock's value that represents a claim on riskless assets is $f := \frac{Re^{r\tau}}{s_\tau}$. As f varies, the value of $\hat{\sigma}$ is altered so that the current volatility is held constant at 40% per annum; i.e., $\hat{\sigma} = 0.4/(1-f)$. The current stock price is \$50, $r = 0.05$, the inflation rate is also 5% per annum, and the option matures 3 months hence. The option's exercise price is denoted by k. The lower bound follows from result II of Lemma 1.

Table II

Upper and Lower Bounds on $VaR(5)$ when the Asset's Diffusion Parameter is Non-Decreasing in the Asset's Value and its Volatility Parameter is Non-Increasing in the Asset's Value. $VaR(5)$ is the negative of the 5% quantile of the asset's return distribution.

$T-t$	Lower and Upper Bounds on $VaR(5)$ (in percent)					
	$[\underline{\sigma}, \overline{\sigma}]$					
	$[0.1, 0.2]$		$[0.25, 0.5]$		$[0.5, 1]$	
1/250	0.995	2.028	2.529	5.079	5.031	10.025
1/50	2.067	4.313	5.362	10.823	10.608	21.007
1/12	3.840	8.452	10.451	21.299	20.475	39.843
1/4	5.564	13.454	16.525	34.131	32.040	60.248
1	6.246	22.042	26.747	57.152	51.446	87.062

The inflation rate is 0.05 per annum. The asset's nominal price dynamic is

$$ds_\tau = 0.1 s_\tau d\tau + \sigma(s_\tau, \tau) s_\tau dB_\tau,$$

with, for all s and t, $z_1(s,t) \geq 0$, $\sigma_1(s,t) \leq 0$ and $\sigma(s,t) \in [\underline{\sigma}, \overline{\sigma}]$. Each pair reports the lower bound on VaR given in result I of Lemma 4 and the upper bound on VaR given in result III of Lemma 4. The length of the investment horizon (measured in years) is $T-t$. The pair in the upper-left-hand corner reports that if the annual volatility is known to be between 10% and 20%, then overnight losses of 0.995% or more will occur at least 5% of the time, while overnight losses of 2.028% or more will occur at most 5% of time.