NORTH-HOLLAND

# An Interesting Matrix Exponent Formula

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We prove one interesting formula of exponent type for noncommutative matrices. This problem arises as a limit of Markov process when considering the limit for continuous time. Denote by $I$ the identity matrix; then our main result is the following:

THEOREM. *Let $A$, $B$ be square matrices such that there exists $\lim_{n\to\infty} A^n = C$. Then*

$$\lim_{n\to\infty}\left(A+\frac{B}{n}\right)^n = C + e^{CBC} - I.$$

*Proof.*

*Step 1.* We first prove the following lemma:

LEMMA. *Let $A$, $B$ be square matrices such that there exists $\lim_{n\to\infty} A^n = C$. Then*

$$\lim_{n\to\infty}\left(A+\frac{\varepsilon}{n}B\right)^n = C + \varepsilon CBC + O(\varepsilon^2). \tag{1}$$

*Proof.* The only term of zero order in $\varepsilon$ is $A^n$, which obviously tends to $C$ when $n \to \infty$. Using the operator norm of matrices, we denote $a = \sup\{\|A\|, \|A^2\|, \|A^3\|, \dots\}$; this number is finite, since $\lim_{n\to\infty} A^n = C$.

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Consider all terms linear in $\varepsilon$. They can be described as all possible words in the alphabet $\{A, B\}$ of length $n$ which have only one letter $B$ and $n-1$ letters $A$. The coefficient of $\varepsilon$ is

$$\sum_{j=1}^{n} A^{j-1}BA^{n-j}. \tag{2}$$

For any small $\delta > 0$ there exists a big enough constant $c_1$ [for example, one can take any $c_1 > (a^2/\delta)\|B\|$] such that for all $n > c_1$ we have $(1/n)\|\sum_{j=1}^{n/c_1} A^{j-i}BA^{n-j}\| < \delta$: this sum contains $n/c_1$ terms, and none of them exceeds $a^2\|B\|$. In the same way we can show that $(1/n)\|\sum_{j>n-n/c_1}^{n} A^{j-1}BA^{n-j}\| < \delta$. Thus the influence of the initial and final terms of the sum will be arbitrarily small.

Consider the rest: $(1/n)\sum_{j>n/c_1}^{n-n/c_1} A^{j-1}BA^{n-j}$. This sum consists of $n(1 - 2/c_1)$ terms. When $n \to \infty$ each term has the same limit, equal to $CBC$, so the limit of this sum is $(1 - 2/c_1)\, CBC$. The choice of $c_1$ is in our hands (it should be big enough), so we can claim that the whole sum (2) tends to $CBC$. The lemma is proved. ∎

*Step* 2. In a similar way one can show that terms quadratic in $\varepsilon$ tend to $CBCBC$. These terms are

$$\frac{\varepsilon^2}{n^2}\sum_{j=1}^{n-1}\sum_{k=1}^{n-j} A^{j-1}BA^{k-1}BA^{n-k-j}.$$

Again for any small $\delta > 0$ there exists a constant $c_2$ (for example one can take $c_2 > 3\|B\|^2a^3/\delta$) such that for all $n > c_2$ we have

$$\frac{1}{n^2}\left\|\sum_{j=1}^{n/c_2}\sum_{k=1}^{n-j} A^{j-1}BA^{K-1}BA^{n-k-j} + \sum_{j=1}^{n-1}\sum_{k=1}^{n/c_2} A^{j-1}BA^{k-1}BA^{n-k-j} + \sum_{j=n-n/c_2}^{n-1}\sum_{k=1}^{n-j} A^{j-1}BA^{k-1}BA^{n-k-j}\right\| < \delta.$$

In other words the coefficient of $\varepsilon^2$ has $O(n^2)$ terms of the form

$$AA\cdots ABA\cdots ABA\cdots AA,$$

but among these terms only $O(n)$ have the property that either the first letter $B$ appears very close to the beginning or to the end, or the two letters $B$ are very close to each other.

Any word from this set can be represented as a cell in an $n \times n$ matrix which lies above diagonal (its row number is the first appearance of letter $B$, its column number the second). There are $n(n-1)/2$ above diagonal cells (words). The claim says that only a small portion of them is close to the diagonal (two letters $B$ are close) or close to the upper right boundary (one of letters $B$ is close to beginning/end of the word).

Consider the general case. Expanding the left hand side of (1) and collecting all terms of degree $d \geqslant 1$ in $\varepsilon$, we have $(\varepsilon^d/n^d)(\cdots)$, where $(\cdots)$ is the sum of all words of length $n$ with $d$ letters $B$ and n-d letters $A$. For any small $\delta > 0$ there exists a constant $c_d$ [for example, one can take $c_d > (d+1)\|B\|^d a^{d+1}/\delta$] such that for all $n > c_d$ one can guarantee that the sum of all terms such that there are two letters $B$ with less than $n/c_d$ letters $A$ between them or there is a letter $B$ closer than $n/c_d$ positions from the beginning/end is smaller than $\delta n^d$. The rest of the proof is the same. Note that we do not require all constants $c_d$ be the same, since this is a formal series.[1]

In a similar way we obtain that the coefficient of $\varepsilon^j$ is

$$(1/j!)CBCB \cdots CBC = (1/j!)(CB)^j C.$$

Thus the limit (1) can be formally rewritten as

$$\lim_{n\to\infty}\left(A + \frac{\varepsilon}{n}B\right)^n = C + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!}(CB)^j C.$$

*Step 3.* Because the matrix $C$ is a projection (i.e. $CC = C$), the last expression can be easily written as

$$\lim_{n\to\infty}\left(A + \frac{\varepsilon}{n}B\right)^n = C + \sum_{j=1}^{j!} (CBC)^j = C + e^{\varepsilon CBC} - I.$$

*Step 4.* Note that the series for the exponent converges uniformly for any complex $\varepsilon$. Therefore for $\varepsilon = 1$ we obtain the final result:

$$\lim_{n\to\infty}\left(A + \frac{B}{n}\right)^n = C + e^{CBC} - I.$$

The theorem is proved. ■

[1]A uniform constant $c$ exists for all $d$ only if $\|B\|$ is small enough.

Using the fact that $C = CC$, one can rewrite the final formula also as $C + e^{CBC} - I = Ce^{CBC}$.

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