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Comment on ‘Non-Linear Value-at-Risk’

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Risk management methods based on Value-at-Risk estimate the lowest quantile of possible profits and losses over a fixed time horizon. To calculate this value there is a need to construct an approximation of the probabilistic distribution of P&L. One of the most popular techniques is based on an assumption that the portfolio value can be expressed as a deterministic function of some basic market parameters. Having a distribution of these parameters one can construct the distribution of the value function. The most popular method is delta approach. Here a first order expansion of the value function is used in order to approximate the distribution at the end of the period. Typically the time period is assumed short, in which case the changes in market parameters are distributed almost normally and under this linear approximation the value of the approximated portfolio is also normally distributed.

Value-at-Risk methods based on a delta approximation can not take into account different forms of convexity. An appropriate solution to this problem is to consider a longer series expansion, for example the so-called delta-gamma approximation. However the delta-gamma approximation loses a very useful property of “delta only” approach – linearity. This linearity property is very convenient computationally, since it guarantees that as soon as the market factors are distributed normally, the resulting changes in the portfolio value are also normally distributed.

Denote by x a vector of n market parameters that can be easily measured and their historical distributions are known. For example stock prices, interest rates, exchange rates. Denoting the calendar time by t we can price a portfolio of assets $V(t, x)$. The Value-at-Risk measures the lowest 1% (sometimes 5%) quantile of the distribution of profits and losses of the fixed portfolio over a fixed time horizon (in banking for example 10 business days).

The standard assumption of this measurement is that over a short time horizon the changes in the market factors Δx are normally distributed. If the value of the portfolio is linear in the market factors then the P&L distribution is normal as well and any quantile can be expressed analytically through its mean and standard deviation. However the assumption of linear dependence is often very restrictive, a higher order approximation is required to reflect convexity.

Consider the value function $V(x, t)$ around the current market values x. As soon as the market changes are small and the function V smooth, we can use the Taylor expansion. However the variable x is stochastic. Thus instead of the standard series we have to use Ito's lemma which mixes first and second derivatives but preserves the right order of the remaining term.

$$\begin{aligned} V(x+\Delta x, t+\Delta t) &= V(x,t) + \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial t}\Delta t \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\Delta x)^2 + o(\Delta t + \Delta x^2). \end{aligned}$$

This is the right first order expansion, as soon as we assume that the stochastic variable x follows some diffusion process.

$$\mathrm{d}x = \mu(x,t)\,\mathrm{d}t + \sigma(x,t)\,\mathrm{d}B.$$

Now we can combine the two expressions obtaining

$$\begin{aligned} V(x+\Delta x, t+\Delta t) - V(x,t) &= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\mu + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}\right)\Delta t \\ &+ \frac{\partial V}{\partial x}\sigma\,\Delta B + o(\Delta t + \Delta x^2). \end{aligned}$$

In a multivariable (but one-factor) case this can be written as

$$\begin{aligned} V(x+\Delta x, t+\Delta t) - V(x,t) &= \left(\frac{\partial V}{\partial t} + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}\mu_i + \frac{1}{2}tr(V''\sigma)\right)\Delta t \\ &+ \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}\sqrt{\sigma_{ii}}\,\Delta B + o(\Delta t + \Delta x^2). \end{aligned}$$

Here σ_{ij} is the covariance between x_i and x_j. Note that this formula is different from the standard (non-stochastic) expansion, but the precision of this formula is eventually dt, since over a small time interval $(\Delta x)^2 \sim \Delta t$. However without the Ito's correction term $\frac{1}{2}tr(V''\sigma)\,\Delta t$, this formula has precision only $(\Delta t)^{0.5}$ and not Δt.

The delta-gamma approach developed in the paper of Britten Jones and Schaefer is based on an approximation of order 1.5 and not 2, see Equations (2.1)–(2.4), (4.2). Thus it is not an expansion of second order, but of the order 1.5 only. A detailed description of different approximation techniques applicable to stochastic variables can be found in Milstein (1995), see pages 27 and 136.

The precise derivation of the second order approximation depends on the assumptions regarding the market factor x. Let us demonstrate this effect with a simple example of an arithmetic Brownian motion. Assume that x is a one-dimensional market factor following

$$\mathrm{d}x = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B,$$

then its value is $x(t + \Delta t) = x(t) + \mu\Delta t + \sigma\Delta B$, where ΔB is distributed $N(0, \sqrt{\Delta t})$. Assume for simplicity that V does not depend on t and is a function of x only, then

$$\begin{aligned} V(x_{t+\Delta t}) - V(x_t) \;=\; & \frac{\partial V}{\partial x}(x_{t+\Delta t} - x_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x_{t+\Delta t} - x_t)^2 \\ & + \frac{1}{6}\frac{\partial^3 V}{\partial x^3}(x_{t+\Delta t} - x_t)^3 + \frac{1}{24}\frac{\partial^4 V}{\partial x^4}(x_{t+\Delta t} - x_t)^4 \\ & + o(x_{t+\Delta t} - x_t)^4. \end{aligned}$$

Note that the market change is (in this case) $x_{t+\Delta t} - x_t = \mu\Delta t + \sigma\Delta B$ and this expression has two parts of different orders of magnitude. For a short time interval we have $\Delta B \sim \sqrt{\Delta t}$. A square root of a small number is much larger than the number itself. Thus we can rewrite the expression above as

$$\begin{aligned} V(x_{t+\Delta t}) - V(x_t) \;=\; & \frac{\partial V}{\partial x}(\mu\Delta t + \sigma\Delta B) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\mu\Delta t + \sigma\Delta B)^2 \\ & + \frac{1}{6}\frac{\partial^3 V}{\partial x^3}(\mu\Delta t + \sigma\Delta B)^3 + \frac{1}{24}\frac{\partial^4 V}{\partial x^4}(\mu\Delta t + \sigma\Delta B)^4 \\ & + o(\mu\Delta t + \sigma\Delta B)^4. \end{aligned}$$

The residual term $o(\mu\Delta t + \sigma\Delta B)^4$ can also be written as $o(\Delta t)^2$, then keeping all terms of order $(\Delta t)^2$ and below (i.e. $(\Delta t)^{1/2}$, Δt, $(\Delta t)^{3/2}$) we obtain:

$$\begin{aligned} V(x_{t+\Delta t}) - V(x_t) \;=\; & \frac{\partial V}{\partial x}(\mu\Delta t + \sigma\Delta B) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\mu\Delta t + \sigma\Delta B)^2 \\ & + \frac{1}{6}\frac{\partial^3 V}{\partial x^3}(3\mu\Delta t\sigma^2\Delta B^2 + \sigma^3\Delta B^3) \\ & + \frac{1}{24}\frac{\partial^4 V}{\partial x^4}(\sigma\Delta B)^4 + o(\Delta t)^2. \end{aligned}$$

This is the true second order approximation (in the case of x following an arithmetic Brownian motion).

In a general case after an appropriate correction, the method developed by Britten Jones and Schaefer can be used. This method should be very efficient for a portfolio with a significant gamma exposure. The main achievement of this work is that the authors have found an analytic expression of the final distribution. It is not normal any more, but a non-central chi-squared. As soon as the form of the distribution is known, we can apply an analytic formula developed by the authors to calculate the lowest quantile. Now it is also clear under what additional assumption the expansion used in Britten Jones and Schaefer (1998) is valid:

$$\frac{\partial^3 V}{\partial x^3} = 0, \frac{\partial^4 V}{\partial x^4} = 0.$$

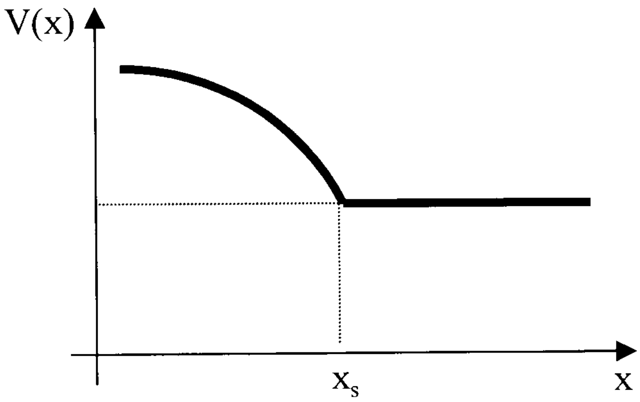

Figure 1. Value of a portfolio using stop-loss order.

For most portfolios including bonds and options the approach suggested by Britten Jones and Schaefer should give a significant advantage because of the analytic nature of the final result. However it is based on one additional assumption – the price function is smooth. To demonstrate why this assumption is important consider two examples. In the first example this approach is applicable and leads to a faster algorithm. In the second one this approach is not applicable (at least directly).

Example 1. Consider a portfolio of loans that can be prepaid. This portfolio is convex by two reasons. First, as any bond portfolio its price is not a linear function of interest rates. Second the prepayment option makes the convexity very important in the area of optimal prepayment. Assuming for simplicity a flat term structure, we can express the value of this portfolio as a smooth function of interest rates. Distribution of interest rates is known, thus one can use a second order approximation for the value of the portfolio and obtain a more precise result than the standard delta approach.

Example 2. Consider a company that uses VaR for risk management of its investment portfolio. Assume that this company uses stop-loss orders to protect against big losses. The value of the portfolio as a function of market parameters will look like one presented in Figure 1.

Note that under the current market conditions the portfolio is a smooth function of the market factor x. However it is convex and not differentiable. Using the delta-gamma approach will not give any advantage and might lead to unrealistic results. Many non financial companies use this type of risk protection. There are well known examples when this strategy did not work appropriately (October 87 crash). But in general this type of loss protection changes the risk profile and makes an analytic approximation useless.

Moreover many companies use instead of a stop loss order a precommitment approach. This approach is used when under some circumstances (defined in advance and not subject to changes) a risky position will be closed. This situation is even more difficult to analyze, since this intention is not reflected in the position at all. However it definitely changes the risk of the portfolio and in my opinion in many cases should be considered as a part of the investment. The critical point here is the precommitment and again an analytical approximation can not be used directly.

References

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