A Note On Integral Representations of the Skorokhod Map

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Abstract

We present a very short derivation of the integral representation of the two-sided Skorokhod reflection Z of a continuous function X of bounded variation, which is a generalization of the integral representation of the one-sided map featured in [1, 4]. We also show that Z satisfies a simpler integral representation when additional conditions are imposed on X.

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1 Introduction

It is known that the one-sided Skorokhod reflection map Z of a continuous real-valued function $X : [0, \infty) \to \mathbb{R}$ of bounded variation satisfies a useful integral representation. This was first shown in Konstantopoulos et al [4] for X belonging to a broad class of continuous functions. Soon afterward, Konstantopoulos and Last [5] showed how to derive an integral representation for Z, when X is allowed to have discontinuity points. These integral representations are also briefly discussed in

Chapter 3 of Baccelli and Brémaud [3], and an interesting result addressing uniqueness of functions satisfying such representations can be found in [1]

The purpose of this note is to present a new, short derivation of the above-mentioned integral representation when X is continuous, that also yields an analogous integral representation for the two-sided Skorokhod map as well. We will also show how our new integral representation simplifies when additional assumptions are placed on the behavior of X.

2 Notation and Main Results

Suppose $A, C : [0, \infty)$ are both nondecreasing and continuous on $[0, \infty)$ and satisfy $A(0) = C(0) = 0$. Using these functions, we define a function $X : [0, \infty) \to \mathbb{R}$ as

$$X(t) = A(t) - C(t)$$

for each $t \geq 0$: clearly X is both continuous and of bounded variation.

Our goal is to derive integral representations for the two-sided Skorokhod reflection $Z : [0, \infty) \to \mathbb{R}$ of X in the interval $[0, a]$, $a > 0$, where Z is a solution from the two-sided Skorokhod problem—see e.g. Kruk et al [6, 7] or Andersen and Mandjes [2]—which is defined as follows.

Theorem 2.1 *There exists a unique pair of cadlag functions Z, Y satisfying*

(i) $Z(t) = X(t) + Y(t) \in [0, a]$ *for each* $t \geq 0$, *and*

(ii) Y *has the decomposition* $Y = L - U$ *as the difference of two increasing cadlag functions* L, U *satisfying*

$$\int_0^\infty \mathbf{1}(Z(t) > 0) dL(t) = 0, \tag{1}$$

$$\int_0^\infty \mathbf{1}(Z(t) < a) dU(t) = 0. \tag{2}$$

The existence and uniqueness of the reflection Z was proven by Tanaka [9]. Closed-form expressions for Z in terms of X are also known: see [2, 6, 7].

We now present our main result, which is an integral representation for the two-sided Skorokhod reflection Z of X.

Theorem 2.2 *For each $t \geq 0$, we have*

$$Z(t) = \int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s) + \int_s^t \mathbf{1}(Z(u) = a)d(A(u) - C(u))\right) dA(s). \tag{3}$$

Observe that when $a = \infty$, the integral representation simplifies to

$$Z(t) = \int_0^t \mathbf{1}(Z(s) > C(t) - C(s))dA(s)$$

which is the integral representation found in [1, 3, 4].

Our proof of Theorem 2.2 will make use of the following simple lemmas. Note also that Z is both continuous on $[0, \infty)$, and of bounded variation on each compact subinterval: this is a consequence of Proposition 1.3 of [6], and we will often make use of this fact in our derivations.

Lemma 2.1 *For $0 \leq s < t$, we have*

$$\int_s^t \mathbf{1}(Z(u) = a)d(A(u) - C(u)) = U(t) - U(s).$$

Proof This follows from both (1) and (2): here

$$\begin{aligned}
\int_s^t \mathbf{1}(Z(u) = a)d(A(u) - C(u)) &= \int_s^t \mathbf{1}(Z(u) = a)d(Z(u) - L(u) + U(u)) \\
&= \int_s^t \mathbf{1}(Z(u) = a)dZ_a(u) - \int_s^t \mathbf{1}(Z(u) = a)dL(u) + \int_s^t \mathbf{1}(Z(u) = a)dU(u) \\
&= \int_{Z(s)}^{Z(t)} \mathbf{1}(x = a)dx + U(t) - U(s) = U(t) - U(s).
\end{aligned}$$

where the last line follows from applying a change-of-variable result, which is valid since Z is continuous and of bounded variation on $[0, t]$: see e.g. the corollary on pg. 42 of Protter [8]. $\diamondsuit$

Lemma 2.2 *For each $t \geq 0$, we have*

$$\int_0^t \mathbf{1}(Z(s) > C(t) - C(s) + U(t) - U(s))dL(s) = 0.$$

Proof Here

$$\begin{aligned}\int_0^t \mathbf{1}(Z(s) > C(t) - C(s) + U(t) - U(s))dL(s) &= \int_0^t \mathbf{1}(Z(s) > C(t) - C(s) + U(t) - U(s)])\mathbf{1}(Z(s) = 0)dL(s) \\ &= \int_0^t \mathbf{1}(0 > C(t) - C(s))\mathbf{1}(Z(s) = 0)dL(s) = 0\end{aligned}$$

since C and U are both nondecreasing functions on $[0, \infty)$. $\diamondsuit$

We are now ready to prove Theorem 2.2. By applying first Lemma 2.1, then Lemma 2.2, then a change-of-variable, we have for each $t \geq 0$

$$\begin{aligned}&\int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s) + \int_s^t \mathbf{1}(Z(u) = a)d(A(u) - C(u))\right) dA(s) \\ = &\int_0^t \mathbf{1}(Z(s) > C(t) - C(s) + U(t) - U(s))dA(s) \\ = &\int_0^t \mathbf{1}(Z(s) + C(s) + U(s) > C(t) + U(t))d(Z(s) + C(s) + U(s) - L(s)) \\ = &\int_0^t \mathbf{1}(Z(s) + C(s) + U(s) > C(t) + U(t))d(Z(s) + C(s) + U(s)) \\ = &\int_0^{Z(t)+C(t)+U(t)} \mathbf{1}(x > C(t) + U(t))dx = Z(t)\end{aligned}$$

which establishes (3).

3 A Simpler Integral Representation

The appearance of the integral term $\int_s^t \mathbf{1}(Z(u) = a)(A - C)(du)$ within the indicator function found in (3) makes using this representation to derive useful versions of Little's law difficult, so one would like to know if the representation can be simplified in some way, in the hopes of finding a representation that is more amenable to computation. We show in this subsection that simplification is possible, once additional restrictions are placed on both A and C.

Throughout this section, we assume $[0, \infty)$ can be written as the union of two Borel measurable sets S_A, S_C in $[0, \infty)$ that satisfy the following properties:

$$\int_{S_A} dC(s) = 0, \qquad \int_{S_C} dA(s) = 0. \tag{4}$$

These conditions are essentially equivalent to saying that the measures induced by both A and C

have disjoint support. When these conditions hold, we observe that the integral representation (3) of Z simplifies considerably, as shown in the following theorem.

Theorem 3.1 *Z satisfies the following integral representation: for each $t \geq 0$,*

$$Z(t) = \int_0^t \mathbf{1}(Z(s) > C(t) - C(s))\mathbf{1}(Z(s) < a)A(ds). \tag{5}$$

We derive this result by making use of a number of simple lemmas. Our first lemma is a slight extension of Lemma 2.1.

Lemma 3.1 *For each arbitrary real number $t \geq 0$, we have for each set $B \in \mathcal{B}([0,t])$ (the Borel measurable subsets of $[0,t]$)*

$$\int_B \mathbf{1}(Z(s) = a)d(A(s) - C(s)) = \int_B dU(s).$$

Proof This statement follows from Lemma 2.1, combined with a standard monotone class argument. $\diamondsuit$

Lemma 3.2 *For each $t \geq 0$, we have*

$$\int_0^t \mathbf{1}(Z_a(s) = a)dC(s) = 0.$$

Proof Using Lemma 3.1, we first observe that

$$\begin{aligned} 0 \leq \int_{S_C \cap [0,t]} dU(s) &= \int_{S_C \cap [0,t]} \mathbf{1}(Z(s) = a)d(A(s) - C(s)) \\ &= \int_{S_C \cap [0,t]} \mathbf{1}(Z(s) = a)dA(s) - \int_{S_C \cap [0,t]} \mathbf{1}(Z(s) = a)dC(s) \\ &= -\int_{S_C \cap [0,t]} \mathbf{1}(Z(s) = a)dC(s) \end{aligned} \tag{6}$$

with the third equality being a consequence of condition (4). Combining both inequality (6) with the fact that C is nondecreasing, we get $\int_{S_C \cap [0,t]} \mathbf{1}(Z(s) = a)dC(s) = 0$, and so

$$\int_0^t \mathbf{1}(Z(s) = a)dC(s) = \int_{S_A \cap [0,t]} \mathbf{1}(Z(s) = a)dC(s) + \int_{S_C \cap [0,t]} \mathbf{1}(Z(s) = a)dC(s) = 0$$

which proves our claim, since (4) implies $\int_{S_A \cap [0,t]} \mathbf{1}(Z(s) = a)dC(s) = 0$. $\diamondsuit$

Lemma 3.3 *For* $0 \le s \le t$,

$$\int_0^t \boldsymbol{1}\left(Z(s) > C(t) - C(s)\right) \boldsymbol{1}\left(Z(s) = a\right) d(Z(s) + C(s)) = 0.$$

Proof Let $s_a = \sup\{s \in [0,t] : C(t) - C(s) \ge a\}$, where $s_a = 0$ when $C(t) < a$. Then by Lemma 3.2, we have

$$\begin{aligned} & \int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s)\right) \mathbf{1}\left(Z(s) = a\right) d(Z(s) + C(s)) \\ = & \int_{s_a}^t \mathbf{1}\left(Z(s) = a\right) d(Z(s) + C(s)) \\ = & \int_{s_a}^t \mathbf{1}\left(Z(s) = a\right) dZ(s) + \int_{s_a}^t \mathbf{1}\left(Z(s) = a\right) dC(s) \\ = & \int_{Z(s_a)}^{Z(t)} \mathbf{1}\left(x = a\right) dx + 0 = 0 \end{aligned}$$

proving our result. $\diamondsuit$

Our final lemma is analogous to Lemma 2.2.

Lemma 3.4 *For each* $t \ge 0$, *we have*

$$\int_0^t \boldsymbol{1}(Z(s) > C(t) - C(s)) \boldsymbol{1}(Z(s) < a) dL(s) = 0.$$

Proof Here

$$\begin{aligned} \int_0^t \mathbf{1}(Z(s) > C(t) - C(s)) \mathbf{1}(Z(s) < a) dL(s) &= \int_0^t \mathbf{1}(Z(s) > C(t) - C(s)) \mathbf{1}(Z(s) = 0) dL(s) \\ &= \int_0^t \mathbf{1}(0 > C(t) - C(s)) \mathbf{1}(Z(s) = 0) dL(s) = 0 \end{aligned}$$

since $C(t) - C(s) \ge 0$ for $0 \le s \le t$. $\diamondsuit$

We are now ready to prove Theorem 3.1. Starting with the right-hand-side of (5), we find that

$$\begin{aligned}
& \int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s)\right) \mathbf{1}\left(Z(s) < a\right) dA(s) \\
= & \int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s)\right) \mathbf{1}\left(Z(s) < a\right) d(Z(s) + C(s) - L(s) + U(s)) \\
= & \int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s)\right) \mathbf{1}\left(Z(s) < a\right) d(Z(s) + C(s)) \\
= & \int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s)\right) d(Z(s) + C(s)) \\
& - \int_0^t \mathbf{1}\left(Z(s) > C(t) - C(s)\right) \mathbf{1}\left(Z(s) = a\right) d(Z(s) + C(s)) \\
= & \int_0^t \mathbf{1}\left(Z(s) + C(s) > C(t)\right) d(Z(s) + C(s)) \\
= & \int_0^{Z(t)+C(t)} \mathbf{1}\left(x > C(t)\right) dx \\
= & \int_{C(t)}^{Z(t)+C(t)} dx = Z(t)
\end{aligned}$$

where the second equality follows from both the definition of U and Lemma 3.4, the fourth equality follows from Lemma 3.3, and the final equality follows from applying a change-of-variable.

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