MORE SOLUTIONS: CHAPTER 6, CLASS OF JULY 29

BENJAMIN YAKIR

Problems, 1. The sample space is composed of 36 points, all occurring with equal probability. Probabilities of events characterized by values of random variables may be obtained by the count of the number of sample points within each event, divided by 36.

(a) The support of (X,Y) is the set $\{(x,y) : 1 \leq x \leq 6, x+1 \leq y \leq 2x\}$. The probabilities are $\mathrm{P}(X = x, Y = y) = 2/36$ if $y \neq 2x$ and $\mathrm{P}(X = x, Y = 2x) = 1/36$.

(b) The support of (X,Y) is the set $\{(x,y) : 1 \leq y \leq 6, x \leq y\}$. The probabilities are $\mathrm{P}(X = x, Y = y) = 1/36$ if $x < y$ and $\mathrm{P}(X = x, Y = x) = x/36$.

(c) The support of (X,Y) is the set $\{(x,y) : 1 \leq y \leq 6, x \leq y\}$. The probabilities are $\mathrm{P}(X = x, Y = y) = 2/36$ if $x < y$ and $\mathrm{P}(X = x, Y = x) = 1/36$.

Problems, 8. (a) Integrating over x and then over y produces:

$$\int_0^\infty \int_{-y}^{y} c(y^2 - x^2)e^{-y} dx dy = c(4/3) \int_0^\infty y^3 e^{-y} dy = c(4/3)\Gamma(4) = 8c \; .$$

The integral of a density equals one, hence $c = 1/8$.

(b) From the integration over x that was conducted in (a) we get that the marginal density of Y is of the structure of the gamma distribution. Therefore, $Y \sim \text{Gamma}(4,1)$. For every fixed real x, y ranges over the interval $|x| < y < \infty$. Integrating the density in that range produces

$$f_X(x) = \int_{|x|}^\infty c(y^2 - x^2)e^{-y} dy = c\Big[- (y^2 - x^2)e^{-y}\Big]_{|x|}^\infty + 2c \int_{|x|}^\infty y e^{-y} dy \; .$$

The evaluation of the tern in square brackets produces zero. The integration of the second term, yet again using integration by parts, leads to

$$f_X(x) = (1/4)(1 + |x|)e^{-|x|} \; .$$

(c) The expectation of X equals zero by symmetry.

Problems, 9. (a) Clearly, the function is non-negative over the range. Integration with respect to y produces $f_X(x) = (6/7)(2x^2 + x)$. Integrating this function with respect to x produces one.

(b) Computed as part of (a). The support is the interval $[0,1]$.

(c) The event $\{X < Y\}$ corresponds to the set $\{(x,y) : 0 \leq x \leq 1, x < y < 2\}$. The probability of the event is the integral of the density over the set:

$$\int_0^1 \int_x^2 \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big) dy dx = \frac{6}{7}\int_0^1 \Big(x(2-x) + \frac{x(4-x^2)}{4}\Big) dx = \frac{6}{7}\Big(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{16}\Big)$$

(d) The probability of the event $\{Y > 1/2, X < 1/2\}$ is:

$$\int_0^{1/2}\int_{1/2}^2 \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)dydx = \frac{6}{7}\int_0^{1/2}\Big(\frac{3x^2}{2} + \frac{15x}{16}\Big)dx = \frac{6}{7}\Big(\frac{1}{16} + \frac{15}{8}\frac{1}{16}\Big)$$

The probability of the event $\{X < 1/2\}$ is:

$$\frac{6}{7}\int_0^{1/2}(2x^2 + x)dx = \frac{6}{7}\Big(\frac{1}{12} + \frac{1}{8}\Big) .$$

Therefore,

$$\mathrm{P}(Y > 1/2|X < 1/2) = \Big(\frac{1}{16} + \frac{15}{8}\frac{1}{16}\Big)\Big/\Big(\frac{1}{12} + \frac{1}{8}\Big) .$$

(e) The expectation of X is:

$$\mathbb{E}(X) = \frac{6}{7}\int_0^1 (2x^3 + x^2)dx = \frac{6}{7}\Big(\frac{1}{2} + \frac{1}{3}\Big) = \frac{5}{7} .$$

(f) The expectation of Y is:

$$\int_0^1\int_0^2 y\frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)dydx = \frac{6}{7}\int_0^1\Big(x^2 + \frac{4x}{3}\Big)dx = \frac{6}{7}\Big(\frac{1}{3} + \frac{2}{3}\Big) = \frac{6}{7}$$

Notice that if we have done first the integration with respect to x than we would have obtained the marginal density of Y, multiplied by y. The integration of the latter with respect to y over the range produces, indeed, the expectation of Y. By Fubini's theorem the order of integration is exchangeable.

Problems, 13. In order to simplify the notation let us denote time as the time after 12:00 in unites of one hour. The marginal distribution of the arrival time of the man (X) is uniform over the interval $[1/4, 3/4]$ and that of the woman (Y) is uniform over the interval $[0, 1]$. By independence, the joint distribution of the arrival times is the product of the two, which produces the uniform distribution over the rectangle $[1/4, 3/4] \times [0, 1]$. Notice that the density is equal to 2 over that range.

They will not have to wait to each other more than 1/12 of an hour is

$$\{|Y - X| \le 1/12\} = \{-1/12 \le Y - X \le 1/12\} = \{X - 1/12 \le Y \le X + 1/12\} .$$

Integrating the density over the appropriate set we get:

$$\mathrm{P}(|Y - X| \le 1/12) = \int_{1/4}^{3/4}\int_{x-1/12}^{x+1/12} 2dydx = 1/6 .$$

The probability that the man arrives first is

$$\mathrm{P}(X < Y) = \int_{1/4}^{3/4}\int_x^1 2dydx = 2\int_{1/4}^{3/4}(1 - x)dx = \frac{1}{2} .$$

(One may also show this result based on considerations of symmetry.)

Problems, 14. Let us take $L = 1$. Let X and Y be two independent U$(0, 1)$. We are interested in the distribution of the random variable $U = |X - Y|$. The support of this random variable is the interval $[0, 1]$. The CDF is given by for any x in the support by:

$$\mathrm{P}(U \le x) = \mathrm{P}(|X-Y| \le x) = \mathrm{P}(Y-x \le X \le Y+x) = 1-2(1-x)^2/2 = 1-(1-x)^2 \ .$$

The computation of the probability of the event is conducted by computing each complement. This complement is a union of two triangles. The density is $f_U(x) = 2(1-x)$, which is the $(Beta)(1, 2)$ density. For other L the distribution is L times a $(Beta)(1, 2)$ random variable.

Problems, 19. The function $f(x, y)$ is non-negative. It integrates over the range to

$$\int_0^1 \int_0^x \frac{1}{x} dy dx = \int_0^1 \frac{x}{x} dx = 1 \ .$$

Hence, it is a density.

(a) Fixing y, we get get that $y \le x \le 1$. Therefore

$$f_Y(y) = \int_y^1 \frac{1}{x} dx = -\log y \ .$$

(b) From the computation above we get that $X \sim \mathrm{U}(0, 1)$.

(c) $\mathbb{E}(X) = 1/2$.

(d) $\mathbb{E}(Y) = -\int_0^1 y \log y dy = \int_0^\infty e^{-x} x e^{-x} dx = (1/4) \int_0^\infty x^{2-1} 2^2 e^{-2x} dx = 1/4$, which results from the change of variable $x = -\log y$, thus $y = e^{-x}$, $dy = -e^{-x} dx$. The range of integration is $(\infty, 0]$, which is the negative of the integral that is taken with $[0, \infty)$ as the range of integration. The final integral is obtained by completion to the Gamma$(2, 2)$ density ($\Gamma(2) = 1$).

Problems, 20. In the first case the two random variables are independent. The marginal density of x is $f_X(x) = xe^{-x}$, for $x > 0$, and $f_Y(y) = e^{-y}$, for $y > 0$. The joint density is obtained as a product of these two marginal densities. In the second case the two random variables are not independent. This can be verified from the examination of the support, which is not consistent with the support independent random variables may share.

Problems, 22. (a) The two random variables are not independent. The support is indeed of an appropriate form. However, $f_X = x + 0.5$, $0 \le x \le 1$ and $f_Y(y) = y + 0.5$, $0 \le y \le 1$, and thus $f(x, y) \ne f_X(x) f_Y(y)$.

(b) $f_X = x + 0.5$, $0 \le x \le 1$

(c)

$$\mathrm{P}(X + Y < 1) = \int_0^1 \int_0^{1-x} (x + y) dy dx = \int_0^1 \left[(1-x)x + \frac{(1-x)^2}{2} \right] dx = \frac{2}{3}$$