MORE SOLUTIONS: CHAPTER 5, CLASS OF JULY 27

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Problems, 16. Let $X \sim N(40, 4^2)$ and $Z \sim N(0,1)$. The probability of having a year with over 50 inches of rain is

$$\mathrm{P}(X > 50) = \mathrm{P}\Big(\frac{X-40}{4} > \frac{50-40}{4}\Big) = \mathrm{P}(Z > 2.5) = 0.0062 \ ,$$

where the last number was taken from the Standard Normal Distribution Table. Hence, the probability of not having that amount of rain for 10 years is $(1 - 0.0062)^{10} = 0.9397 \approx 94\%$. Apart from the assumed normality we assume that the probability of the occurrence this extreme event is independent among years.

Theoretical, 2. Start with the case where $Y \geq 0$ is a non-negative random variable. By definition,

$$\mathbb{E}[Y] = \int_0^\infty y f_Y(y) dy \ .$$

Notice that $f_Y(y) = -\frac{\partial}{\partial y}(1 - F_Y(y))$. Hence, applying integration by parts, we get

$$\int_0^\infty y f_Y(y) dy = \Big[-y(1 - F_Y(y))\Big]_0^\infty + \int_0^\infty (1 - F_Y(y)) dy \ .$$

However, if $\mathbb{E}[Y] = \int_0^\infty y f_Y(y) dy < \infty$ then the monotone sequence of the tail converges to zero: $\lim_{y\to\infty} \int_y^\infty x f_Y(x) dx = 0$. This implies, since $y(1 - F_Y(y)) < \int_y^\infty x f_Y(x) dx$, that evaluation of the term in square brackets at $y = \infty$ is zero. Clearly, the evaluation at $y = 0$ is also zero, hence

$$\int_0^\infty y f_Y(y) dy = \int_0^\infty (1 - F_Y(y)) dy = \int_0^\infty P(Y > y) dy \ .$$

For a general random variable we may use the representation

$$Y = \max\{Y, 0\} - \max\{-Y, 0\}$$

and apply the previous result to each of the non-negative random variables in the difference.

Theoretical, 5. Using the hint we get:

$$\int_0^\infty \mathrm{P}(X^n > t) dt = \int_0^\infty \mathrm{P}(X^n > x^n) n x^{n-1} dx = \int_0^\infty \mathrm{P}(X > x) n x^{n-1} dx \ ,$$

since the change of variable did not change the limit of integration and since $dt = n x^{n-1} dx$. The lest equality follows from the fact that $\{X^n > x^n\} = \{X > x\}$ are the same event.

Theoretical, 11. Let $Z \sim N(0,1)$ and let g be a differentiable function. Consider the integral

$$\mathbb{E}[g'(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g'(z) e^{-\frac{1}{2}z^2} dz \ .$$

The derivative of the function $\exp\{-z^2/2\}$ is $-z \exp\{-z^2/2\}$. Using integration by parts we get the relation

$$\int_{-\infty}^{\infty} g'(z) e^{-\frac{1}{2}z^2} dz = \Big[g(z) e^{-\frac{1}{2}z^2}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(z) \big[-z e^{-\frac{1}{2}z^2}\big] dz \ .$$

If the function g is such that the limits, at $\pm\infty$, of the term in the large square brackets converges to zero then we get that

$$\int_{-\infty}^{\infty} g'(z) e^{-\frac{1}{2}z^2} dz = \int_{-\infty}^{\infty} z g(z) e^{-\frac{1}{2}z^2} dz \ ,$$

which implies the relation $\mathbb{E}[g'(Z)] = \mathbb{E}[Zg(Z)]$. Part (b) follows from the given relation by taking $g(z) = z^n$. (Notice that the condition holds.) Part (c) can be solved by applying (b) with $n = 3$ to get $\mathbb{E}[Z^4] = 3\mathbb{E}[Z^2] = 3$.