MORE SOLUTIONS: CHAPTER 3, CLASS OF JULY 15

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Problems, 74. Denote an experiment to be "A roles a pair of dice and then B roles a pair of dice". Let E be the event {A obtained a sum of 9} and let F be the event {A did not obtained a sum of 9 and B obtained a sum of 6}. The E and F are mutually exclusive. Thus, based on Problem 3.36, one obtains that E occur before F in probability $\mathrm{P}(E)/[\mathrm{P}(E)+\mathrm{P}(F)]$. Now, since $\mathrm{P}(E) = 4/36$ and $\mathrm{P}(F) = (1-4/36)(5/36)$ we get that the probability in question is 9/19.

Problems, 86. The experiment involves selecting two subsets, A and B, independent of each other and with equally-likely probabilities. The sample space is composed of pairs (A, B), with the probability 2^{-2n} for each pair. Summing the probabilities of the points in the event $\{A \subset B\}$ we get

$$\mathrm{P}(A \subset B) = \sum_{\{B: B\subset S\}} \sum_{\{A: A\subset B\}} 2^{-2n} = \sum_{i=0}^{n} \binom{n}{i} 2^i 2^{-2n} = (2+1)^n 2^{-2n} = (3/4)^n ,$$

where the first equation following from the fact that there are $\binom{n}{i}$ ways of selecting a subset B of size i and each such subset has 2^i subsets A. The third equation follows from the Binomial Formula.

A and B are disjoint if, and only if, $A \subset B^c$. The same argument given above, but this time summing over B^c, will produce the conclusion that $\mathrm{P}(A \cap B = \emptyset) = (3/4)^n$.

Theoretical, 18. Let E be the event that no run of consecutive heads appears in the n toeses of a fair coin and let G_i, $i = 1, \ldots, n$ be the events that the first tail took place in the i-th trial. Observe that the G events are a partition of the sample space. Hence, by the Complete Probability Formula

$$Q_n = \mathrm{P}(E) = \sum_{i=1}^{n} \mathrm{P}(E|G_i)\mathrm{P}(G_i) .$$

However, once i is larger than 3 we know the three consecutive heads already occur, with the corollary that the conditional probability of E is zero. For $i = 1, 2, 3$, the event E will be materialized if, and only if, there will be a run of three heads in the remaining $n - i$ trails. By independence, since the outcomes in the first i trial do not affect those of the remaining $n - i$ trials, these conditional probabilities are equal to Q_{n-i}. The given formula is now straightforward.

Now,

$$\begin{aligned}
Q_4 =& 0.5 \cdot 1 + 0.25 \cdot 1 + 0.125 \cdot 1 = 0.875 \\
Q_5 =& 0.5 \cdot 0.875 + 0.25 \cdot 1 + 0.125 \cdot 1 = 0.8125 \\
Q_6 =& 0.5 \cdot 0.8125 + 0.25 \cdot 0.875 + 0.125 \cdot 1 = 0.75 \\
Q_7 =& 0.5 \cdot 0.75 + 0.25 \cdot 0.8125 + 0.125 \cdot 0.875 = 0.6875 \\
Q_8 =& 0.5 \cdot 0.6875 + 0.25 \cdot 0.75 + 0.125 \cdot 0.8125 = 0.6328125
\end{aligned}$$

Theoretical, 28. Consider two dice. Let E_1 be the event that the first dice produces 3 and let E_2 be the event that the second dice produces 2. The two events are independent and have probability of $1/6$ each. Let F be the event that the sum of the two dice is 6. Then the conditional probability of the intersection of the two events is zero but there marginal probabilities are positive. Consequently, the two events cannot be independent in the conditional distribution.