MORE SOLUTIONS: CHAPTER 3, CLASS OF JULY 14

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Problems, 57. (a) The stock will be back in the original price after two days of trading if either it went up in the first day and down in the second or down in the first and up in the second. The probability of the first case is $p \times (1-p)$, since the movements are independent. The probability of the second possibility is $(1-p) \times p$. Thus, the total probability of the event is $2p(1-p)$.

(b) In a total of three days it should move up twice and down once. There are three different combinations by which this can happen. The probability of each combination is $p^2(1-p)$. As a result, the probability we compute is $3p^2(1-p)$.

(c) The probability that it went up in the first day and ended up after three days with value of one is $2p^2(1-p)$, since there are two combinations in (b) that correspond to this situation. In order to obtain the conditional probability, given the event in (b), we divide this probability, which is the probability of the intersection, with the probability from (b) to obtain the conditional probability of 2/3.

Problems, 66. Let E_1 and E_2 be two independent events with probabilities p_1 and p_2, respectively. For future reference notice that

$$\mathrm{P}(E_1 \cup E_2) = \mathrm{P}(E_1) + \mathrm{P}(E_2) - \mathrm{P}(E_1 \cap E_2) = p_1 + p_2 - p_1 p_2 \; . \tag{0.1}$$

(a) Let E_i correspond to the event that i-th relay is closed. For the current to flow the event E_5 must occur and also either both E_1 and E_2 or both of E_3 and E_4. The event of a closed circuit is $\big((E_1 \cap E_2) \cup (E_3 \cap E_4)\big) \cap E_5$. Using independence and relation (0.1) produces the probability

$$\mathrm{P}\big(\big\{(E_1 \cap E_2) \cup (E_3 \cap E_4))\big\} \cap E_5\big) = (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5 \; .$$

(b) Consider the same events as in (a). Notice that over E_3 it is sufficient to have either E_1 or E_2, together with either E_4 or E_5. Over E_3^c one must have either both E_1 and E_4 or both E_2 and E_5. Consequently, we have a close circuit over the union of the disjoint events $\big((E_1 \cup E_2) \cap (E_4 \cup E_5)\big) \cap E_3$ and $\big((E_1 \cap E_4) \cup (E_2 \cap E_5)\big) \cap E_3^c$. The probability of the first event is

$$\mathrm{P}\big(\big\{(E_1 \cup E_2) \cap (E_4 \cup E_5)\big\} \cap E_3\big) = (p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5)p_3 \; .$$

The probability of the event event is

$$\mathrm{P}\big(\big\{(E_1 \cap E_4) \cup (E_2 \cap E_5)\big\} \cap E_3^c\big) = (p_1p_4 + p_2p_5 - p_1p_2p_4p_5)(1 - p_3) \; .$$

The probability of the event in question is the sum of these two probabilities.

Theoretical, 15. Consider the given probability in the context of a random walk in which success corresponds to a move up and a failure to a move to the left. In each step the former takes place in probability p and the later in probability $1-p$. Steps are independent of each other. Observe that the event under consideration corresponds to all path that reach level r for the first time after n steps. All these paths are characterized by the fact that they reach the point $(n-r, r-1)$ after $n-1$ steps and go up in the last step. All such paths have probability $p^r(1-p)^{n-r}$, since they go up r times and go to the left $n-r$ times. The number of such paths, which is the number of paths that pass through the point $(n-r, r-1)$ is $\binom{n-1}{r-1}$, which produces the claim.