**STAT 4015 Q**
Final, August 13, 2009, 6:15pm - 7:50pm
(Solutions)

**Problems 1.** A box contains 21 balls, 7 of which are black and the others are white. Three of the balls are removed from the box and are replaced by six black balls. Afterwards, four more balls are randomly chosen from the box.

(1) Compute the probability that at least one of the four balls chosen in the second stage was white.
(2) Compute the probability that all three balls selected at the first stage where black, given that at least one of the four balls chosen in the second stage was white.

**Solution.** Denote by $E$ the given event. Going over the different possibilities for the number of black balls in the first draw we get that

$$\mathrm{P}(E) = \sum_{x=0}^{3} \left[ \frac{\binom{7}{x}\binom{21-7}{3-x}}{\binom{21}{3}} \left(1 - \frac{\binom{7-x+6}{4}}{\binom{24}{4}}\right)\right]$$

Let now $F_3$ be the event that all balls in the first stage were black. Then

$$\mathrm{P}(F_3|E) = \mathrm{P}(F_3 \cap E)/\mathrm{P}(E) = \left[ \frac{\binom{7}{3}\binom{21-7}{0}}{\binom{21}{3}} \left(1 - \frac{\binom{7-3+6}{4}}{\binom{24}{4}}\right)\right]/\mathrm{P}(E)$$

**Problems 2.** Let the density of $X$ be $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$ (and zero otherwise).

(1) Show that $f_X(x)$ is the conditional density of $X - a$, given $\{X \geq a\}$, for all non-negative $a$.
(2) The decay time of two radioactive isotopes, measured in second, has the given density, with $\lambda = 1$ for the one and $\lambda = 2$ for the other. A material is composed of a mixture of 50% the first isotope and 50% the second (in mole, namely the number of atoms). How many seconds would elapse until half the radionuclide's atoms in the material decay?

**Solution.** Clearly, the support is $x \geq 0$. The Probability of the event $\{X - a > x\}$ is $e^{-\lambda(x+a)}$. The Probability of the event $\{X \geq a\}$ is $e^{-\lambda a}$. Hence, the conditional CDF of $X - a$, given the event, is

$$1 - \mathrm{P}(X - a > x | X \geq a) = 1 - e^{-\lambda(a+x)}/e^{-\lambda a} = 1 - e^{-\lambda x} \ .$$

Taking derivative with respect to $x$ produces $\lambda e^{-\lambda x}$ as the conditional density.

Let $x$ be the half time. It is found by solving the equation

$$\frac{1}{2} = \frac{1}{2}e^{-x} + \frac{1}{2}e^{-2x}$$

Let $y = e^{-x}$. The equation becomes $y^2 + y - 1 = 0$. The solutions are $y_{1,2} = (-1 \pm \sqrt{5})/2$, of which only $y = (\sqrt{5} - 1)/2 \approx 0.618$ is non-negative. Consequently, $x \approx -\log(0.618) = 0.48$.

**Problems 3.** A joint density is given by $f_{\mathrm{X,Y}}(x, y) = cy^3 xe^{-2x}$, $0 < x < \infty$, $0 < y < x$.

(1) Evaluate $c$.
(2) Identify the marginal distribution of $X$.
(3) Computed the covariance between $X$ and $Y$.

**Solution.** Integrating with respect to $y$ we get that

$$f_X(x) = cxe^{-2x}\frac{x^4}{4} = \frac{c}{4}x^{6-1}e^{-2x} ,$$

for $x > 0$. It follows that $X \sim \text{Gamma}(6, 2)$. Consequently, $c/4 = 2^6/5!$ and thus $c = 2^8/5!$. The expectation of $X$ is $\mathbb{E}(X) = 6/2 = 3$. The expectation of $Y$ is

$$\mathbb{E}(Y) = \int_0^\infty \int_0^x ycy^3 xe^{-2x} dydx = \frac{c}{5}\int_0^\infty x^6 e^{-2x} dx = \frac{c}{5}6!\,2^{-7} = 12/5 .$$

The joint moment is

$$\mathbb{E}(XY) = \int_0^\infty \int_0^x xycy^3 xe^{-2x} dydx = \frac{c}{5}\int_0^\infty x^7 e^{-2x} dx = \frac{c}{5}7!\,2^{-8} = 42/5 .$$

The covariance is $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 42/5 - 3 \cdot 12/5$.

**Problems 4.** A random graph with $n$ nodes is formed by the inclusions of each of the $\binom{n}{2}$ edges with probability $p$, independently of the other edges. A complete sub-graph is a (non-empty) sub-collection of nodes with the property that all connecting edges are included. Let $X$ be the number of complete sub-graphs in a random graph.

(1) Compute $\mathbb{E}(X)$.
(2) Compute $\mathrm{P}(X > 0)$.

**Solution.** Consider all $2^n - 1$ non-empty subsets of nodes. Let $i$ be the index of such subset and $X_i$ the indicator of the associated sub-graph being complete. Note that $X = \sum_i X_i$. For any $i$ with $k$ nodes we have that

$$\mathbb{E}(X_i) = \mathrm{P}(X_i = 1) = p^{\binom{k}{2}} ,$$

since all $\binom{k}{2}$ edges must be present. Consequently

$$\mathbb{E}(X) = \sum_{k=1}^{n} \binom{n}{k} p^{\binom{k}{2}} .$$

One there is an edge in the graph one may find a complete graph; the one composed of the two nodes and the given edge. Hence, the event of having no connected graph coincides with the event of having no edge. The latter event occurs in probability $(1 - p)^n$. Hence, $\mathrm{P}(X > 0) = 1 - (1 - p)^n$.

**Problems 5.** The density of $X$ is $f_X(x) = c(x + 2)^3(3 - x)^2$, for $-2 < x < 3$.

(1) Compute $\mathbb{E}(X)$ and $\text{Var}(X)$.
(Hint: notice that $X$ is a linear transformation of a beta random variable.)
(2) Let $X_1, X_2, \ldots, X_{25}$ be a random sample from the distribution of $X$. Denote by $\bar{X}_{25}$ the sample average. Approximate the probability $\mathrm{P}\big(\big|\bar{X}_{25}\big| < 0.5\big)$.

**Solution.** Notice that $X = 5Y - 2$, for $Y \sim \text{Beta}(4,3)$. Recall that the expectation of is $4/7$ and the variance is $4 \cdot 3/(7^2 \cdot 8) = 0.0306$. It follows $\mathbb{E}(X) = 0.857$ and $\text{Var}(X) = 0.765$. The standard deviation of the average is $\sqrt{0.765/25} = 0.175$. Hence, by the CLT,

$$\text{P}\big(|\bar{X}_{25}| < 0.5\big) = \text{P}\big(-0.5 < \bar{X}_{25} < 0.5\big) = \text{P}\big(-7.754 < Z_{25} < -2.04\big) \approx 0.0207 \,.$$