MORE SOLUTIONS: CHAPTER 7, CLASS OF AUGUST 4

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Problems, 75. The moment generating function of X is that of a Poisson(2) distribution and that of Y corresponds to the B(10, 3/4) distribution. Accordingly, the two random variables have the given distributions. They are also independent.

(a) Considering the three possibilities:

$$\begin{aligned} \mathrm{P}(X+Y=2) &= \mathrm{P}(X=0,Y=2)+\mathrm{P}(X=1,Y=1)+\mathrm{P}(X=2,Y=2) \\ &= e^{-2}\Big[\binom{10}{2}\frac{3^2}{4^{10}}+\binom{10}{1}\frac{2\cdot 3}{4^{10}}+\binom{10}{0}\frac{2^2}{2\cdot 4^{10}}\Big] . \end{aligned}$$

(b) The product equals zero if either of the random variables is:

$$\begin{aligned} \mathrm{P}(XY=0) &= \mathrm{P}(\{X=0\}\cup\{Y=0\}) \\ &= \mathrm{P}(X=0)+\mathrm{P}(Y=0)-\mathrm{P}(X=0,Y=0) \\ &= e^{-2}+\binom{10}{0}\frac{1}{4^{10}}-e^{-2}\binom{10}{0}\frac{1}{4^{10}} . \end{aligned}$$

(c) By independence, $\mathbb{E}(XY)=\mathbb{E}(X)\cdot\mathbb{E}(Y)=2\cdot 10\cdot 3/4$.

Theoretical, 19. By bi-linearity, $\mathrm{Cov}(X+Y,X-Y)=\mathrm{Cov}(X,X)-\mathrm{Cov}(X,Y)+\mathrm{Cov}(Y,X)-\mathrm{Cov}(Y,Y)=\mathrm{Var}(X)-\mathrm{Var}(Y)$. If X and Y share the same marginal distribution then their variances are also equal and the last difference becomes zero.

Theoretical, 24. For any t, $0<\mathbb{E}[(X-tY)^2]=\mathbb{E}(X^2)-2t\mathbb{E}(XY)+t^2\mathbb{E}(Y^2)$, with equality holding only if $\{tY=X\}$ with probability one. Take $t=\mathbb{E}(XY)/\mathbb{E}(Y^2)$ to obtain $0<\mathbb{E}(X^2)-\mathbb{E}(XY)^2/\mathbb{E}(Y^2)$, hence the proof.

A corollary of this inequality, applied to the centered random variables $X-\mathbb{E}(X)$ and $Y-\mathbb{E}(Y)$ is the claim that $[\mathrm{Cov}(X,Y)]^2\le\mathrm{Var}(X)\mathrm{Var}(Y)$ (with equality holding only if $Y=aX+b$ for some constants a and b). Consequently, the correlation between the two random variables, $\rho(X,Y)=\mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$, falls inside the interval $[-1,1]$.

Theoretical, 35. Splitting the event according to the values of X we get from the formula of complete probability that

$$\mathrm{P}(X<Y)=\sum_{x=0}^{\infty}\mathrm{P}(X<Y|X=x)\mathrm{P}(X=x)=\sum_{x=0}^{\infty}\mathrm{P}(x<Y)\mathrm{P}(X=x) ,$$

where we illuminated the conditioning since the events $\{X=x\}$ and $\{Y>x\}$ are independent. For a geometric random variable with parameter p one has that $\mathrm{P}(Y>x)=(1-p)^x$, since the event occurs if, and only if, the first x trails ended in failure. For $p=1-s\Rightarrow 1-p=s$ we obtain the representation.

Theoretical, 41. (a) The random variables X and Y are not independent since the conditional distribution of Y, given $\{X = x\}$ may obtain only two values, x and $-x$, and thus depends on the value that X obtains and cannot be equal to the marginal distribution (which is normal).

(b) Y and I are independent since the conditional distribution of Y, given $\{I = 1\}$, is standard normal. Since the distribution of $-X$ is also standard normal it follows that the conditional distribution of Y, given $I = 0$, is again standard normal.

(c) The marginal distribution of Y is standard normal since

$$\begin{aligned}\mathrm{P}(Y \leq y) &= \mathrm{P}(Y \leq y | I = 1)/2 + \mathrm{P}(Y \leq y | I = 0)/2 \\ &= \mathrm{P}(X \leq y | I = 1)/2 + \mathrm{P}(-X \leq y | I = 0)/2 \\ &= \Phi(y)/2 + \Phi(y)/2 = \Phi(y) \ .\end{aligned}$$

(c) Since the expectation of X is zero the covariance is equal to the mixed second moment:

$$\mathbb{E}(XY) = \frac{1}{2}\int_{-\infty}^{\infty} x \cdot x \cdot \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx + \frac{1}{2}\int_{-\infty}^{\infty} x \cdot (-x) \cdot \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx = 0 \ .$$

(Notice that the joint distribution of I and X has a support over the two parallel lines in the plain, one line associated with the value zero of I and the other with the value one.)

Theoretical, 49. Let $Y = \sigma Z + \mu$, $Z \sim \mathrm{N}(0, 1)$, be a normal random variable and let $X = e^Y$ be the log-normal random variable. Hence,

$$\mathbb{E}(X) = \mathbb{E}[e^Y] = M_{\mathrm{Y}}(1) = e^{\mu + \sigma^2/2} \ .$$

Likewise,

$$\mathbb{E}(X^2) = \mathbb{E}[e^{2Y}] = M_{\mathrm{Y}}(2) = e^{2\mu + 2^2\sigma^2/2} \ .$$

Consequently,

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right) \ .$$

Theoretical, 50. Let $g(t) = \log M(t)$. Recall that $M(0) = 1$, $M'(0) = \mathbb{E}(X)$, and $M''(0) = \mathbb{E}(X^2)$. Taking a derivative with respect to t we get that

$$g'(t) = \frac{M'(t)}{M(t)} \Rightarrow g'(0) = \frac{M'(0)}{M(0)} = \mathbb{E}(X) \ .$$

Taking a second derivative we get that

$$g''(t) = \frac{M''(t)}{M(t)} - \frac{[M'(t)]^2}{[M(t)]^2} \Rightarrow g''(0) = \frac{M''(0)}{M(0)} - \frac{[M'(0)]^2}{[M(0)]^2} = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \ .$$

Theoretical, 54. The expectation of Z is zero. Therefore, $\mathrm{Cov}(Z, Z^2) = \mathbb{E}[Z \cdot Z^2] = \mathbb{E}[Z^3] = 0$. The last equality follows from the fact that $g(z) = z^3 e^{-z^2/2}/\sqrt{2\pi}$ is anti-symmetric ($g(-z) = -g(z)$) so its integral over the real line is equal to zero.