MORE SOLUTIONS: CHAPTER 7, CLASS OF AUGUST 3

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Problems, 4. (a) Integrating over x and then over y produces the joint moment: $\mathbb{E}(XY) = \int_0^1 \int_0^y x \cdot y \cdot (1/y) dx dy = \int_0^1 y^2/2 dy = 1/6$.

(b) Likewise, $\mathbb{E}(X) = \int_0^1 \int_0^y x \cdot (1/y) dx dy = \int_0^1 y/2 dy = 1/4$ and

(c) $\mathbb{E}(Y) = \int_0^1 \int_0^y y \cdot (1/y) dx dy = \int_0^1 y dy = 1/2$.

Problems, 5. Assuming continuity, we get that the travelling distance in the horizontal direction is $X \sim \mathrm{U}(0, 3/2)$ and the travelling distance in the vertical direction is $Y \sim \mathrm{U}(0, 3/2)$. The expected travel distance to the accident is $\mathbb{E}(X + Y) = 3/2$.

Problems, 9. (a) Let X_i be the indicator of the event that the i-th urn is empty, for $1 \leq i \leq n$. $T = \sum_{i=1}^n X_i$ total number of empty urns. $\mathbb{E}(T) = \sum_{i=1}^n \mathbb{E}(X_i) = n\mathrm{P}(X_1 = 1)$. The probability that the first urn is empty after the first ball is $(1 - 1/n)$, and after the last ball is placed it is $\mathrm{P}(X_1 = 1) = (1 - 1/n)^n$. Consequently, $\mathbb{E}(T) = n(1 - 1/n)^n$.

(b) This problem can be associated with the problem of splinting the integer n into $r = n$ integers that sum to n. The number of points in the resulting sample space is $\binom{2n-1}{n}$, with steps to the right indicating balls and steps up indicating the boundaries between urns. The event of having no empty urns corresponds to the constraint $n_i \geq 1$, $1 \leq i \leq r$. However, there is only one path that obey this constraint, having one ball in each urn, hence the probability is $\binom{2n-1}{n}^{-1}$.

Problems, 38. The joint moment is:

$$\mathbb{E}(XY) = \int_0^\infty \int_0^x x \cdot y \cdot (2e^{-2x}/x) dy dx = \int_0^\infty x^2 e^{-2x} dx = \frac{\Gamma(3)}{2^3} = \frac{3}{4}$$

and the expectations are:

$$\mathbb{E}(X) = \int_0^\infty \int_0^x x \cdot (2e^{-2x}/x) dy dx = \int_0^\infty 2x e^{-2x} dx = \frac{\Gamma(2)}{2} = \frac{1}{2}$$

and

$$\mathbb{E}(Y) = \int_0^\infty \int_0^x y \cdot (2e^{-2x}/x) dy dx = \int_0^\infty x e^{-2x} dx = \frac{\Gamma(2)}{4} = \frac{1}{4} \, .$$

It follows that $\mathrm{Cov}(X, Y) = 3/4 - 1/8 = 5/8$.

Problems, 47. (a) The marginal distribution of D_i, which counts the number of edges that are actually present among the potential $n-1$ edges that connects i to the other vertices has the $\mathrm{B}(n-1,p)$ distribution.

(b) Consider D_i and D_j. There is the potential edge that connects i to j and then there are the n_-2 potential edges that connect i to the other vertices and the $n-2$ potential edges that connect j to the other vertices. The presence or absence of this edges are independent events. it turns out that one can represent $D_i = X_i + Y$ and $D_j = X_j + Y$, where X_i, X_j have both the $\mathrm{B}(n-2,p)$ distribution and $Y\ sim\mathrm{B}(1,p)$, all independent of each other. It follows that

$$\mathrm{Cov}(D_i, D_j) = \mathrm{Cov}(X_i + Y, X_j + Y) = \mathrm{Cov}(Y, Y) = p(1-p)\ .$$

Hence, the correlation, which is the covariance divided by the product of the standard deviations, is $1/(n-1)$.

Problems, 56. Denote by X the number of people that enter the elevator at the ground floor and let us compute the probability that no one gets off the elevator at the first floor. Denote this event by E_1. Dividing the sample space according to the number of people that enter the elevator we get for the formula of complete probability that

$$\mathrm{P}(E_1) = \sum_{x=0}^{\infty} \mathrm{P}(E_1 | X = x)\mathrm{P}(X = x) = \sum_{x=0}^{\infty} \Big(\frac{N-1}{N}\Big)^x e^{-10}\frac{10^x}{x!} e^{-10+\frac{N-1}{N}10} = e^{-\frac{10}{N}}\ .$$

Denote E_i to be the parallel events for the other stories and T to be the the number of occurrences of these events we get that $\mathbb{E}(T) = Ne^{-\frac{10}{N}}$.

As a matter of fact, one can show that the number of people that leave at each floor is a Poisson random variable with expectation $10/N$. and that the numbers are independent between floors. This produces another way of proving the claim.