בס״ד

$$Var(\bar{y}) = Cov(\bar{y}, \bar{y})$$

$$= Cov\left(\tfrac{1}{n} \Sigma_i I_i Y_i,\ \tfrac{1}{n} \Sigma_j I_j Y_j\right)$$

$$= \left(\tfrac{1}{n}\right)^2 \Sigma_i \Sigma_j Y_i Y_j \, Cov(I_i, I_j)$$

$$= \left(\tfrac{1}{n}\right)^2 \Sigma_i \Sigma_j Y_i Y_j \left(\pi_{ij} - \pi_i \pi_j\right)$$

$$= \left(\tfrac{1}{n}\right)^2 \Sigma_i \Sigma_j Y_i Y_j \pi_{ij} - \left(\tfrac{1}{n}\right)^2 \Sigma_i \Sigma_j Y_i Y_j \pi_i \pi_j$$

$$= \mathrm{I} - \mathrm{II}$$

$$\mathrm{II} = \left(\tfrac{1}{n} \Sigma_i \pi_i Y_i\right)\left(\tfrac{1}{n} \Sigma_i \pi_j Y_j\right) = \bar{Y}^2$$

$$\mathrm{I} = \left(\tfrac{1}{n}\right)^2 \Sigma_i \frac{n}{N} Y_i^2 + \left(\tfrac{1}{n}\right)^2 \Sigma_{i \neq j} Y_i Y_j \frac{n(n-1)}{N(N-1)}$$

$$= \left(\tfrac{1}{n}\right)^2 \Big\{ \frac{n}{N} \Sigma_i Y_i^2 - \frac{n(n-1)}{N(N-1)} \Sigma_i Y_i^2$$

$$+ \frac{n(n-1)}{N(N-1)} \Sigma_i \Sigma_j Y_i Y_j \Big\}$$

$$= \left(\tfrac{1}{n}\right)^2 \Big\{ \frac{n}{N}\left(1 - \frac{n-1}{N-1}\right) \Sigma_i Y_i^2$$

$$+ \frac{n(n-1)}{N(N-1)} N^2 \bar{Y}^2 \Big\}$$

$$= \left(\tfrac{1}{n}\right)^2 \left(\frac{n}{N}\right) \frac{N-n}{N-1} \Sigma_i Y_i^2 + \left(\tfrac{1}{n}\right)^2 \frac{n(n-1)}{N(N-1)} N^2 \bar{Y}^2$$

$$Var(\bar{y}) = \left(\frac{1}{n}\right)^2 \frac{\cancel{n}}{N} \frac{N-n}{N-1} \sum Y_i^2$$

$$+ \left(\frac{1}{n}\right)^2 \frac{\cancel{n}}{N} \frac{n-1}{N-1} N^2 \bar{Y}^2 - \bar{Y}^2$$

$$= \left(\frac{1}{n}\right)\left(\frac{1}{N}\right) \left[\frac{N-n}{N-1} \sum Y_i^2 + \frac{n-1}{N-1} N^2 \bar{Y}^2 - nN\bar{Y}^2 \right]$$

$$\frac{n-1}{N-1} N^2 \bar{Y}^2 - nN\bar{Y}^2$$

$$= \bar{Y}^2 \left[\frac{N^2(n-1)}{N-1} - \frac{nN(N-1)}{N-1} \right]$$

$$= \frac{\bar{Y}^2}{N-1} \left[\cancel{N^2 n} - N^2 - \cancel{nN^2} + nN \right]$$

$$= \frac{\bar{Y}^2}{N-1} \left[-N^2 + nN \right]$$

$$= \frac{\bar{Y}^2}{N-1} \left[-N(N-n) \right]$$

Thus

$$Var(\bar{y}) = \left(\frac{1}{n}\right)\left(\frac{1}{N}\right) \left[\frac{N-n}{N-1} \sum Y_i^2 - \frac{N-n}{N-1} N\bar{Y}^2 \right]$$

$$= \left(\frac{1}{n}\right)\left(\frac{N-n}{N-1}\right)\left(\frac{1}{N}\right) \left[\sum (Y_i - \bar{Y})^2 \right]$$

$$= \left(\frac{1}{n}\right)\left(1 - \frac{n}{N}\right) S^2 = (1-f) S^2$$

with $S^2 = \frac{1}{N-1} \sum (Y_i - \bar{Y})^2$

$f = \frac{n}{N}$

$1 - f = f.p.c.$

$f = 1$ if $n = N$
$f \approx 0$ if $n \ll N$