Post-stratification = simple random sample + break up into strata after the fact

Some properties:

1) $[n_1, n_2, \cdots, n_L]$ is now a random vector

$$Pr(n_1 = m_1, \cdots, n_L = m_L) = \frac{\binom{N_1}{m_1} \cdots \binom{N_L}{m_L}}{\binom{N}{n}}$$

Pf: denominator = total number of possible samples overall

numerator = number of possible samples satisfying the condition

2) Conditional on $\{n_1 = m_1, \cdots, n_L = m_L\}$, we have stratified random sampling with sample sizes $m_1, \cdots, m_L$. That is

a) Simple random sample of size m_h in Stratum h

b) Strata independent of each other

Pf: We have $Pr(S^\circ = S) = \binom{N}{n}^{-1} \quad \forall S \in \mathcal{S}$.

Denote $\mathcal{S}(m_1, \cdots, m_L)$ the set of samples with sizes $m_1, \cdots, m_L$ in the respective strata. We have, for any $S \in \mathcal{S}(m_1, \cdots, m_L)$

$$Pr(S^\circ = S \mid n_1 = m_1, \cdots, n_L = m_L)$$

$$= Pr(\{S^\circ = S\} \cap \{n_1 = m_1, \cdots, n_L = m_L\}) / Pr(n_1 = m_1, \cdots, n_L = m_L)$$

$$= Pr(S^\circ \in S) / Pr(n_1 = m_1, \cdots, n_L = m_L)$$

[since $S^\circ = S$ implies $\{n_1 = m_1, \cdots, n_L = m_L\}$]

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using Property ①

$$= \left[1 / \binom{N}{n} \right] / \left[\binom{N_1}{m_1} \cdots \binom{N_L}{m_L} / \binom{N}{n} \right]$$

$$= \binom{N_1}{m_1}^{-1} \cdots \binom{N_L}{m_L}^{-1} . \qquad (*)$$

Now with simple random sampling of a sample S_h^o of size h from Stratum h, we have

$$\Pr(S_h^o = S_h) = \binom{N_h}{m_h}^{-1}$$

by standard random sampling theory.

The expression $(*)$ is the product of these probabilities, which proves the claim.

3) If $N_1, \cdots, N_L$ are all large, then

$$\Pr(n_1 = m_1, \cdots, n_L = m_L) \doteq \frac{n!}{m_1! \cdots m_L!} W_1^{m_1} \cdots W_L^{m_L}$$

<u>Pf</u>: We have

$$\Pr(n_1 = m_1, \cdots, n_L = m_L)$$

$$= \binom{N_1}{m_1} \cdots \binom{N_L}{m_L} \Big/ \binom{N}{n}$$

$$= \left[\frac{N_1!}{m_1!(N_1-m_1)!} \cdots \frac{N_L!}{m_L!(N_L-m_L)!}\right] \Big/ \left(\frac{N!}{n!(N-n)!}\right]$$

$$= \frac{n!}{m_1! \cdots m_L!} \left[\frac{N_1!}{(N_1-m_1)!} \cdots \frac{N_L!}{(N_L-m_L)!}\right] \Big/ \left[\frac{N!}{(N-n)!}\right]$$

$$= \frac{n!}{m_1! \cdots m_L!} \{ [N_1(N_1-1)\cdots(N_1-m_1+1)]$$

$$[N_2(N_2-1)\cdots(N_2-m_2+1)]$$

$$\cdots [N_L(N_L-1)\cdots(N_L-m_L+1)\}$$

$$/ [N(N-1)\cdots(N-n+1)]$$

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$$= \frac{n!}{m_1! \cdots m_L!}$$

$$\left\{ \left[\frac{N_1}{N} \; \frac{N_1 - 1}{N} \; \cdots \; \frac{N_1 - m_1 + 1}{N} \right] \right.$$

$$\left[\frac{N_2}{N} \; \frac{N_2 - 1}{N} \; \cdots \; \frac{N_2 - m_2 - 1}{N} \right]$$

$$\left. \cdots \left[\frac{N_L}{N} \; \frac{N_L - 1}{N} \; \cdots \; \frac{N_L - m_L + 1}{N} \right] \right\}$$

$$\Big/ \left\{ \frac{N}{N} \; \frac{N-1}{N} \; \cdots \; \frac{N - n + 1}{N} \right\} .$$

Now, if the N_h are all large, then

$$\frac{N_h - l}{N} \doteq \frac{N_h}{N} = W_h , \quad l = 1, \cdots, m_L - 1,$$

and similarly $\frac{N - l}{N} \doteq 1, \quad l = 1, \cdots, n.$

So we get

$$Pr(n_1 = m_1, \cdots, n_L = m_L)$$

$$\doteq \left(\frac{n!}{m_1! \cdots m_L!} \right)$$

$$\{ \underbrace{[W_1 \cdots W_1]}_{m_1 \text{ times}} \underbrace{[W_2 \cdots W_2]}_{m_2 \text{ times}} \cdots \underbrace{[W_L \cdots W_L]}_{m_L \text{ times}} \}$$

$$= \frac{n!}{m_1! \cdots m_L!} W_1^{m_1} \cdots W_L^{m_L}$$

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4) $\Pr(n_h = m_h) = \binom{N_h}{m_h}\binom{N-N_h}{n-m_h} \Big/ \binom{N}{n}$

$$E[n_h] = n\frac{N_h}{N} = nW_h$$

$$Var(n_h) = \frac{N_h(N-N_h)\,n(N-n)}{N^2(N-1)}$$

$$= n\left(\frac{N-n}{N-1}\right)W_h(1-W_h)$$

For $\omega_h = n_h/n$,

$$E[\omega_h] = W_h$$

$$Var(\omega_h) = \frac{1}{n}\,\frac{N-n}{N-1}\,W_h(1-W_h)$$

For N_h large

$$n_h \dot{\sim} Bin(n, W_h)$$