בס"ד

טענה: $\Pr(\chi^2_{2D} > 2\mu) = \Pr(Poi(\mu) < D)$

הוכחה:

צפיפות של χ^2_{2D}

$$\Pr(\chi^2_{2D} > 2\mu)$$

$$= \int_{2\mu}^{\infty} \frac{x^{\frac{1}{2}(2D-2)} e^{-\frac{1}{2}x}}{2^{2D/2}\,\Gamma(2D/2)}\,dx$$

$$= \int_{2\mu}^{\infty} \frac{x^{D-1} e^{-1/2 x}}{2^{D}\,(D-1)!}\,dx$$

כאן נבצע אינטגרציה בחלקים

$$\int v\,du = uv - \int u\,dv$$

אם

$$du = e^{-1/2 x}$$
$$u = -2e^{-1/2 x}$$
$$v = x^{D-1}/(2^D (D-1)!)$$
$$dv = (D-1)\,x^{D-2}/(2^D(D-1)!)$$
$$= x^{D-2}/(2^D(D-2)!)$$

נקבל

$$\Pr(\chi^2_{2D} > 2\mu)$$

$$= \left.\frac{-x^{D-1} e^{-\frac{1}{2}x}}{2^{D-1}(D-1)!}\right|_{2\mu}^{\infty} + \int_{2\mu}^{\infty} \frac{x^{D-2} e^{-\frac{1}{2}x}}{2^{D-1}(D-2)!}\,dx$$

$$= \frac{(2\mu)^{D-1} e^{-\frac{1}{2}(2\mu)}}{2^{D-1}(D-1)!} + \int_{2\mu}^{\infty} \frac{x^{D-2} e^{-1/2 x}}{2^{D-1}(D-2)!}\,dx$$

$$= \frac{\mu^{D-1} e^{-\mu}}{(D-1)!} + \int_{2\mu}^{\infty} \frac{x^{D-2} e^{-\frac{1}{2}x}}{2^{D-1}(D-2)!}\,dx$$

$$= \Pr(Poi(\mu) = D-1) + \int_{2\mu}^{\infty} \frac{x^{D-2} e^{-1/2 x}}{2^{D-1}(D-2)!}\,dx$$

בס"ד

קיבלנו

$$\Pr(\chi^2_{2D} > 2\mu)$$
$$= \Pr(Poi(\mu) = D-1) + \int_{2\mu}^{\infty} \frac{x^{D-2} e^{-\frac{1}{2}x}}{2^{D-1}(D-2)!}\, dx$$

כעת, אם נחזור על הפעולות לעיל פעם שנייה (אינטגרציה בחלקים וכו') נקבל

$$\Pr(\chi^2_{2D} > 2\mu)$$
$$= \Pr(Poi(\mu) = D-1) + \Pr(Poi(\mu) = D-2)$$
$$+ \int_{2\mu}^{\infty} \frac{x^{D-3} e^{-\frac{1}{2}x}}{2^{D-2}(D-3)!}\, dx$$

ואם נחזור על התהליך שוב ושוב נקבל בסופו של דבר

$$\Pr(\chi^2_{2D} > 2\mu) = \Pr(Poi(\mu) = D-1) + \Pr(Poi(\mu) = D-2)$$
$$+ \cdots + \Pr(Poi(\mu) = 0)$$
$$= \Pr(Poi(\mu) < D)$$

where the coefficients a_i depend only on ε (or the significance level $1-\varepsilon$). By employing Dwyer's (1951) method of pivotal condensation to solve the normal equations obtained from (18.21) for $\nu = 1(1)30$ and the tabled values of $\chi^2_{\nu,\varepsilon}$ for $\varepsilon = 0.95$, 0.99 and 0.999, Gilbert determined the following values for the coefficients a_i:

	SIGNIFICANCE LEVEL, $1-\varepsilon$		
	0.05	0.01	0.001
a_0	2.518232	5.174627	9.205913
a_1	1.282189	1.402766	1.542498
a_2	−0.00211427	−0.00303260	−0.00411568
a_3	1.371169	1.858993	2.314035

Gilbert has discussed the maximum error incurred while using the interpolation formula in (18.21); he has shown that this interpolation method provides quite reasonable values (often correct to two decimal places or more) for ν small and even nonintegral for the values of ε considered.

5 APPROXIMATIONS AND COMPUTATIONAL ALGORITHMS

As mentioned earlier in Section 3, the standardized χ^2_ν distribution tends to the unit normal distribution as $\nu \to \infty$ [see (18.12)]. The simple approximation obtained from (18.12) given by

$$F_{\chi^2_\nu}(x) \approx \Phi\left(\frac{x-\nu}{\sqrt{2\nu}}\right), \tag{18.22}$$

however, is not very accurate unless ν is rather large. Better approximations may be obtained by using the asymptotic normality of various functions of χ^2_ν, even though only approximate standardization is effected. Among the best-known simple approximations are Fisher's (1922) approximation given by

$$F_{\chi^2_\nu}(x) \approx \Phi(\sqrt{2x} - \sqrt{2\nu - 1}) \tag{18.23}$$

and the Wilson-Hilferty (1931) approximation given by

$$F_{\chi^2_\nu}(x) \approx \Phi\left(\sqrt{\frac{9\nu}{2}}\left\{\left(\frac{x}{\nu}\right)^{1/3} - 1 + \frac{2}{9\nu}\right\}\right). \tag{18.24}$$

Of these two approximations the second one is definitely more accurate, but both the approximations are better than the one in (18.22). From the approximations in (18.23) and (18.24), one may obtain approximations to the

percentage point $\chi^2_{\nu,\varepsilon}$ as

$$\chi^2_{\nu,\varepsilon} \approx \frac{1}{2}\left(U_\varepsilon + \sqrt{2\nu-1}\right)^2, \tag{18.25}$$

$$\chi^2_{\nu,\varepsilon} \approx \nu\left(\sqrt{\frac{2}{9\nu}}\,U_\varepsilon + 1 - \frac{2}{9\nu}\right)^3 \tag{18.26}$$

respectively; here U_ε denotes $\Phi^{-1}(\varepsilon)$, the lower ε percentage point of the standard normal distribution. It should be mentioned that the addition of $(U_\varepsilon^2 - 1)/6$ to (18.25) will make it very nearly equal to the usually better approximation in (18.26), unless ν is small or ε is close to 0 or 1. This point is illustrated by the values in Table 18.1.

Table 18.1 Comparison of Approximations to χ^2 Percentile Points

			Approximation		Difference	
ε	ν	$\chi^2_{\nu,\varepsilon}$	(18.25)	(18.26)	(18.26)–(18.25)	$\frac{1}{6}(U_\varepsilon^2 - 1)$
0.01	5	0.5543	0.2269	0.5031	0.2762	
	10	2.5582	2.0656	2.5122	0.4466	0.7353
	25	11.5240	10.9215	11.4927	0.5712	
0.05	5	1.1455	0.9182	1.1282	0.3100	
	10	3.9403	3.6830	3.9315	0.2485	0.2843
	25	14.6114	14.3388	14.6086	0.2698	
0.10	5	1.6103	1.4765	1.6098	0.1333	
	10	4.8652	4.7350	4.8695	0.1345	0.1071
	25	16.4734	16.3503	16.4788	0.1285	
0.50	5	4.3515	4.5000	4.3625	−0.1375	
	10	9.3418	9.5000	9.3480	−0.1520	−0.1667
	25	24.3366	24.5000	24.3392	−0.1608	
0.90	5	9.2364	9.1658	9.2078	0.0420	
	10	15.9872	15.9073	15.9677	0.0604	0.1071
	25	34.3816	34.2920	34.3701	0.0781	
0.95	5	11.0705	10.7873	11.0439	0.2666	
	10	18.3070	18.0225	18.2918	0.2693	0.2843
	25	37.6525	37.3667	37.6452	0.2785	
0.99	5	15.0862	14.1850	14.4599	0.2749	
	10	23.2093	22.3463	23.2393	0.8930	0.7353
	25	44.3141	43.4904	44.3375	0.8471	