phenomenon under study. Given this inference, one may predict the value of the odds ratio in a new population and test the difference between the observed and predicted values; one may predict the effects on mortality of controlling the factor represented by x (in our example, an air pollutant); and one may of course predict the effects on mortality of controlling smoking habits.

The representation (5.30) of the logarithm of ω suggests that the logarithm of the sample odds ratio,

$$L = \ln(o), \tag{5.31}$$

is an important measure of association. Natural logarithms are tabulated in Table A.5. The standard error of L has been studied by Woolf (1955), Haldane (1956), and Gart (1966). A better estimate of $\ln(\omega)$ was found to be

$$L' = \ln(o'), \tag{5.32}$$

where o' is defined in (5.20), and a good estimate of its standard error was found to be

$$\text{s.e.}(L') = \sqrt{\frac{1}{n_{11}+.5} + \frac{1}{n_{12}+.5} + \frac{1}{n_{21}+.5} + \frac{1}{n_{22}+.5}}\ . \tag{5.33}$$

When the logistic model of (5.25) and (5.26) obtains, $\ln(\omega)$ is seen by (5.30) to be completely independent of x. Even if, instead, a model specified by a cumulative normal distribution is assumed, $\ln(\omega)$ is nearly independent of x (Edwards, 1966; Fleiss, 1970). The logistic model is far more manageable for representing rates and proportions than the cumulative normal model, however, and has been so used by Bartlett (1935), Winsor (1948), Dyke and Patterson (1952), Cox (1958, 1970), Grizzle (1961, 1963), Maxwell and Everitt (1970), and Fienberg (1977).

5.5. TESTING HYPOTHESES ABOUT THE ODDS RATIO

Theoretical result. Let attention be restricted to fourfold tables with marginal frequencies $n_{1.}$, $n_{2.}$, $n_{.1}$, and $n_{.2}$ fixed at the values actually observed, and suppose that the value of the underlying odds ratio is equal to ω. The expected cell frequencies N_{ij} associated with ω are such that (a) they are consistent with the original data in the sense that they recapture the marginal frequencies:

Table 5.4. Expected frequencies in a fourfold table

Factor A	Factor B		
	Present	Absent	Total
Present	N_{11}	N_{12}	$n_{1.}$
Absent	N_{21}	N_{22}	$n_{2.}$
Total	$n_{.1}$	$n_{.2}$	$n_{..}$

and (b) they are consistent with the value ω in the sense that

$$\frac{N_{11}N_{22}}{N_{12}N_{21}} = \omega. \tag{5.34}$$

The hypothesis that the value of the underlying odds ratio is equal to ω may be tested by referring the value of

$$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\left(|n_{ij} - N_{ij}| - \frac{1}{2}\right)^2}{N_{ij}} \tag{5.35}$$

to the chi square distribution with one degree of freedom. The form of the test statistic in (5.35) is identical to the form of the classic statistic presented in (2.7), and the interpretations of the N_{ij}'s as expected cell frequencies associated with hypothesized values of the odds ratio [$\omega = 1$ in (2.7), ω arbitrary in (5.35)] are also identical.

When ω is the hypothesized value of the odds ratio, and when $\omega \neq 1$, the expected cell frequencies may be found as follows. Define

$$X = \omega(n_{1.} + n_{.1}) + (n_{2.} - n_{.1}) \tag{5.36}$$

and

$$Y = \sqrt{X^2 - 4n_{1.}n_{.1}\omega(\omega - 1)}\ ; \tag{5.37}$$

then

$$N_{11} = \frac{X - Y}{2(\omega - 1)}, \tag{5.38}$$

$$N_{12} = n_{1.} - N_{11}, \tag{5.39}$$

$$N_{21} = n_{.1} - N_{11}, \tag{5.40}$$

and

$$N_{22} = n_{.2} - n_{1.} + N_{11}. \tag{5.41}$$

The following result was proved by Stevens (1951) and by Cornfield (1956). When the marginal frequencies are held fixed and when ω is the value of the odds ratio, n_{ij} (for any one of the four cells) is approximately normally distributed with mean N_{ij} and standard error $1/\sqrt{W}$, where

$$W = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{1}{N_{ij}} \tag{5.42}$$

and the N_{ij}'s are defined by (5.38)–(5.41).

Application. Consider testing the hypothesis that the value of the odds ratio underlying the data of Table 5.1 is $\omega = 5$. The value of X in (5.36) is

$$X = 5(50 + 25) + (150 - 25) = 500, \tag{5.43}$$

and that of Y in (5.37) is

$$Y = \sqrt{500^2 - 4 \times 50 \times 25 \times 5 \times 4} = 387.30. \tag{5.44}$$

The four expected cell frequencies are presented in Table 5.5. Note that

Table 5.5. Expected frequencies for data in Table 5.1 when the odds ratio is 5

Maternal Age	Birthweight B	Birthweight $\bar{B}$	Total
A	14.1	35.9	50
$\bar{A}$	10.9	139.1	150
Total	25	175	200

the odds ratio for the expected frequencies is

$$\frac{14.1 \times 139.1}{35.9 \times 10.9} = 5.0. \tag{5.45}$$

The value of chi square in (5.35) is

$$\chi^2 = \frac{(|10-14.1|-0.5)^2}{14.1} + \frac{(|40-35.9|-0.5)^2}{35.9} + \frac{(|15-10.9|-0.5)^2}{10.9} + \frac{(|135-139.1|-0.5)^2}{139.1} = 2.56, \tag{5.46}$$

and so the hypothesis that $\omega = 5$ is not rejected.

Another test of this hypothesis may be based on results given in the preceding section. If $\lambda = \ln(\omega)$, the quantity

$$\chi^2 = \frac{(L' - \lambda)^2}{(\text{s.e.}(L'))^2} \tag{5.47}$$

may be referred to the chi square distribution with one degree of freedom. For the data at hand,

$$\lambda = \ln(5) = 1.61, \tag{5.48}$$

$$L' = \ln\frac{10.5 \times 135.5}{40.5 \times 15.5} = .82, \tag{5.49}$$

and

$$\text{s.e.}(L') = \sqrt{\frac{1}{10.5} + \frac{1}{40.5} + \frac{1}{15.5} + \frac{1}{135.5}} = .438. \tag{5.50}$$

The value of the chi square statistic in (5.47) is then

$$\chi^2 = \frac{(.82 - 1.61)^2}{.438^2} = 3.25, \tag{5.51}$$

which is larger than the value of the statistic in (5.46) but still indicates a nonsignificant difference from $\omega = 5$.

The value of the chi square statistic in (5.47) based on the log odds ratio usually exceeds that of the statistic in (5.35) based on a comparison of the n_{ij}'s with the N_{ij}'s, but the difference is small when the marginal frequencies are large. If the statistic in (5.35) were defined without the continuity correction, its value would be close to that of the statistic in (5.47) even for moderate sample sizes (see Problem 5.5).

The procedure described in this section is more complicated than the one based on the log odds ratio, but is more accurate. It should be used whenever a hypothesized value of ω is tested.

5.6. CONFIDENCE INTERVALS FOR THE ODDS RATIO

A $100(1-\alpha)\%$ confidence interval for ω may be constructed as follows. The interval consists of all those values of ω for which, when the N_{ij}'s are the associated expected cell frequencies from (5.38)–(5.41),

$$\chi^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\left(|n_{ij} - N_{ij}| - \frac{1}{2}\right)^2}{N_{ij}} \leqq c^2_{\alpha/2}. \tag{5.52}$$

The upper and lower limits are those for which the value of χ^2 equals $c^2_{\alpha/2}$.

The statistic in (5.52) depends on ω not explicitly but only implicitly through (5.36)–(5.41). The criterion for finding the upper and lower limits is therefore not simple. However, it is not overly complicated and can be implemented as follows.

The lower confidence limit, say ω_L, is associated with values of N_{11} and N_{22} smaller than n_{11} and n_{22} and with values of N_{12} and N_{21} larger than n_{12} and n_{21}. The continuity correction in (5.52) is then such that the χ^2 criterion simplifies to, say,

$$\chi^2_L = \left(n_{11} - N_{11} - \tfrac{1}{2}\right)^2 W = c^2_{\alpha/2}, \tag{5.53}$$

where W is defined in (5.42).

The lower limit, ω_L, will have been found when

$$F = \left(n_{11} - N_{11} - \tfrac{1}{2}\right)^2 W - c^2_{\alpha/2} \tag{5.54}$$

is equal to zero. Define

$$T = \frac{1}{2(\omega-1)^2}\left(Y - n_{..} - \frac{\omega-1}{Y}\left[X(n_{1.} + n_{.1}) - 2n_{1.}n_{.1}(2\omega-1)\right]\right), \tag{5.55}$$

$$U = \frac{1}{N_{12}^2} + \frac{1}{N_{21}^2} - \frac{1}{N_{11}^2} - \frac{1}{N_{22}^2}, \tag{5.56}$$

and

$$V = T\left[\left(n_{11} - N_{11} - \tfrac{1}{2}\right)^2 U - 2W\left(n_{11} - N_{11} - \tfrac{1}{2}\right)\right]. \tag{5.57}$$

Let $\omega_L^{(1)}$ be a first approximation to ω_L, let X and Y be the corresponding solutions of (5.36) and (5.37), and let N_{11}, N_{12}, N_{21}, and N_{22} be the corresponding solutions of (5.38)–(5.41). If the value of F in (5.54) is not equal to zero, a second, better approximation to ω_L is

$$\omega_L^{(2)} = \omega_L^{(1)} - \frac{F}{V}. \tag{5.58}$$

If the value of F associated with the second approximation is still not zero (say, if its absolute value exceeds .05), the process has to be repeated.

Convergence to ω_U, the upper confidence limit, proceeds by exactly the same process, except that the continuity correction is taken as $+\frac{1}{2}$ in (5.53), (5.54), and (5.57). Good first approximations to ω_L and ω_U are provided by the limits of the interval based on the log odds ratio,

$$\omega_L^{(1)} = \text{antilog}\left[L' - c_{\alpha/2}\text{s.e.}(L')\right] \tag{5.59}$$

and

$$\omega_U^{(1)} = \text{antilog}\left[L' + c_{\alpha/2}\text{s.e.}(L')\right]. \tag{5.60}$$

Consider again the data of Table 5.1, and suppose that a 95% confidence interval is desired for ω. From (5.49) and (5.50),

$$\begin{aligned} \omega_L^{(1)} &= \text{antilog}(.82 - 1.96 \times .438) \\ &= \text{antilog}(-.04) = .96 \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} \omega_U^{(1)} &= \text{antilog}(.82 + 1.96 \times .438) \\ &= \text{antilog}(1.68) = 5.37. \end{aligned} \tag{5.62}$$

Consider first the lower confidence limit. The value of X in (5.36) associated with $\omega_L^{(1)} = .96$ is

$$X = .96(50 + 25) + (150 - 25) = 197.00 \tag{5.63}$$