Notes on Brownian Motion and Brownian Bridge

The aim of this set of notes is to summarize some basic properties of the Brownian motion and Brownian bridge processes. For more details, consult stochastic process texts such as Cox and Miller's *The Theory of Stochastic Processes*, Freedman's *Brownian Motion and Diffusion*, and Billingley's *Convergence of Probability Measures*.

Brownian motion (BM) is defined as a stochastic process $\{B(t), t \geq 0\}$ with the following properties:

1. $B(0) = 0$.

2. For $t_1 < t_2$, $B(t_2) - B(t_1) \sim N(0, t_2 - t_1)$.

3. For any $t_1 < t_2 < \ldots < t_K$, the random variables $B(t_k) - B(t_{k-1}), k = 2, \ldots, K$ are independent (independent increments property).

4. With probability one, the function $B(t)$ is continuous in $t$.

The Brownian motion process is also known as the Wiener process, after the mathematician Norbert Wiener, who pioneered the mathematical theory of this process.

The above defining properties of the BM process imply that, for any $t_1 < t_2 < \ldots < t_K$, the random vector $(B(t_1), \ldots, B(t_K))$ has a mean-zero multivariate normal distribution, with $\text{Cov}(B(t), B(t)) = \min\{s, t\}$.

Brownian bridge (BB) is defined as a stochastic process $\{B^\circ(u), u \in [0, 1]\}$ with the following properties:

1. $B^\circ(0) = B^\circ(1) = 0$

2. For any $u_1 < u_2 < \ldots < u_K$, the random vector $(B^\circ(u_1), \ldots, B^\circ(u_K))$ has a mean-zero multivariate normal distribution.

3. $\text{Cov}(B^\circ(u), B^\circ(u)) = \min\{u, v\} - uv$.

4. With probability one, the function $B^\circ(u)$ is continuous in $u$.

Below are some elementary properties of the above processes.

1. If $\{B(t)\}$ is a BM and $A$ is any positive number, then $\tilde{B}(t) = A^{1/2}B(t/A)$ is also a BM.

2. If $\{B(t)\}$ is a BM, then $\tilde{B}^\circ(u) = B(u) - uB(1), u \in [0,1]$, is a BB.

3. If $\{B^\circ(u)\}$ is a BB, then $\tilde{B}(t) = (t+1)B^\circ(t/(t+1))$ is a BM.

4. If $\{B(t)\}$ is a BM and $T$ is a fixed positive number, then $\tilde{B}(t) = B(T+t) - B(T)$ is also a BM. Also, $\tilde{B}(t)$ is independent of $\{B(s), s \in [0,T]\}$.

These properties are proven by showing that $\tilde{B}(t)$ satisfies the definition of BM and $\tilde{B}^\circ(t)$ satisfies the definition of BB. The second part of Property 4 follows from the independent increments property of BM.

Now let $B(t)$ be a BM and $B^\circ(u)$ a BB. Let $\Phi$ denote the standard normal distribution function and put $\bar{\Phi} = 1 - \Phi$. We then have

$$\Pr\left(\max_{0\le s\le t} B(s) > a\right) = 2\Pr(B(t) > a) = 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right), \tag{1}$$

$$\Pr\left(\max_{0\le s\le t} |B(s)| > a\right) = 1 - \sum_{k=-\infty}^{\infty} (-1)^k[\Phi((2k+1)a\sqrt{t}) - \Phi((2k-1)a\sqrt{t})], \tag{2}$$

$$\Pr\left(\max_{0\le u\le \kappa} B^\circ(u) > a\right) = \bar{\Phi}\left(\frac{a}{\sqrt{\kappa(1-\kappa)}}\right) + e^{-2a^2}\bar{\Phi}\left(\frac{a(1-2\kappa)}{\sqrt{\kappa(1-\kappa)}}\right), \tag{3}$$

$$\Pr\left(\max_{0\le u\le \kappa} |B^\circ(u)| > a\right) = 2\,\bar{\Phi}\left(\frac{a}{\sqrt{\kappa(1-\kappa)}}\right) - 2\sum_{k=1}^{\infty}(-1)^k e^{-2k^2a^2}[\bar{\Phi}(r(2k-d) - \bar{\Phi}(r(2k+d))], \tag{4}$$

with $r = a\sqrt{(1-\kappa)/\kappa}$ and $d = (1-\kappa)^{-1}$. The two results for BM can be found in stochastic process texts such as those mentioned above. The two results for BB are taken from Hall and Wellner (1980, *Biometrika*), where they are derived from some additional related results for BM.

The key idea in the proofs is an argument known as the "reflection principle." We illustrate by discussing the proof of (1), which is fairly simple. We begin with the fact (stated above) that for any fixed $T > 0$, the process $\tilde{B}_T(u) = B(T+u) - B(T)$ is independent of $\{B(s), s \in [0,T]\}$ and is itself a BM. Moreover, the same holds in the situation where $T$ is a special type of random variable known as a "stopping time." The random variable $T$ is a stopping time (with respect to $\{B(t)\}$) if the occurrence or non-occurrence of the event $\{T \leq t\}$ can be determined from the observations $\{B(s), s \in [0,t]\}$. The extension of the above-mentioned fact to the case where $T$ is a random stopping time is not trivial, but it can be proved. The extension includes the result that $\tilde{B}_T(u)$ is independent of the value of $T$ itself.

In particular, let $T$ be the first time the process $B(\cdot)$ crosses the level $a$. It is clear that $T$ is a stopping time. Now, if $\max_{0\leq s\leq t} B(s) > a$, then $B(\cdot)$ must cross the level $a$ at least once in the interval $[0,t]$, and thus the event $\{\max_{0\leq s\leq t} B(s) > a\}$ is equivalent to the event $\{T \leq t\}$. Now, we can write

$$B(t) = B(T) + [B(t) - B(T)] = a + [B(t) - B(T)].$$

Now, as discussed above, $\tilde{B}_T(u) = B(T+u) - B(T)$ is a BM. Thus, for any $\tau$, the conditional distribution of $B(t) - B(T)$ given $T = \tau$ is $N(0, (t-\tau)^2)$, which is symmetric about zero. It follows that

$$\Pr\left(\max_{0\leq s\leq t} B(s) > a\right) = \Pr(T \leq t) = 2\Pr(T \leq t, [B(t) - B(T)] > 0) = 2\Pr(B(t) > a),$$

as claimed.

The probabilities $\Pr(\max_{0\leq s\leq t} B(s) > a)$ and $\Pr(\max_{0\leq u\leq \kappa} B^{\circ}(u) > a)$ are easy to calculate from the standard normal distribution function. The following two pages give tables of the distribution of $\Pr(\max_{0\leq s\leq 1} |B(s)| > a)$ and $\Pr(\max_{0\leq u\leq \kappa} |B^{\circ}(u)| > a)$ for various values of $\kappa$ (in these pages, BM is denoted $W(x)$ and BB is denoted $W^0(x)$).

## Appendix 2: Calculation of the asymptotic null-distributions

For the test statistics discussed in this paper we have to consider four types of asymptotic null distributions. Tables of critical points are presented in Tables 8–11.

(i) The probability

$$\Pr\left(\sup_{0\leq x\leq 1} |W(x)| > y\right)$$

(Table 8) can be calculated by using the formulae of Anderson (1960). Tables are also provided in the papers by Rényi (1953) and Walsh (1962).

**Table 8**

*Critical points of the distribution of*

$$\sup_{0\leq x\leq 1} |W(x)|$$

| $\Pr\left(\sup_{0\leq x\leq 1} \|W(x)\| > y\right)$ | y | $\Pr\left(\sup_{0\leq x\leq 1} \|W(x)\| > y\right)$ | y |
|---|---|---|---|
| 0·01 | 2·807 | 0·35 | 1·356 |
| 0·02 | 2·577 | 0·40 | 1·281 |
| 0·03 | 2·433 | 0·45 | 1·213 |
| 0·04 | 2·326 | 0·50 | 1·149 |
| 0·05 | 2·241 | 0·55 | 1·089 |
| 0·06 | 2·170 | 0·60 | 1·032 |
| 0·07 | 2·108 | 0·65 | 0·977 |
| 0·08 | 2·054 | 0·70 | 0·924 |
| 0·09 | 2·005 | 0·75 | 0·871 |
| 0·10 | 1·960 | 0·80 | 0·816 |
| 0·15 | 1·780 | 0·85 | 0·759 |
| 0·20 | 1·645 | 0·90 | 0·696 |
| 0·25 | 1·534 | 0·95 | 0·617 |
| 0·30 | 1·439 | | |

**Table 9**

*Critical points of the distribution of*

$$\sup_{0 \leq x \leq \kappa} |W^0(x)|$$

*for selected values of* $\kappa$ *and*

$$h^*(y, \kappa) = \Pr\left(\sup_{0 \leq x \leq \kappa} |W^0(x)| > y\right)$$

| $h^*(y, \kappa)$ | $\kappa = 0{\cdot}1$ | $\kappa = 0{\cdot}2$ | $\kappa = 0{\cdot}3$ | $\kappa = 0{\cdot}4$ | $\kappa = 0{\cdot}5$ | $\kappa = 0{\cdot}6$ | $\kappa = 0{\cdot}7$ | $\kappa = 0{\cdot}8$ | $\kappa = 0{\cdot}9$ | $\kappa = 1{\cdot}0$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·8513 | 1·1513 | 1·3420 | 1·4700 | 1·5518 | 1·5996 | 1·6215 | 1·6273 | 1·6277 | 1·6277 |
| 0·02 | 0·7823 | 1·0592 | 1·2378 | 1·3593 | 1·4382 | 1·4860 | 1·5095 | 1·5167 | 1·5174 | 1·5174 |
| 0·03 | 0·7394 | 1·0026 | 1·1731 | 1·2896 | 1·3676 | 1·4154 | 1·4403 | 1·4488 | 1·4491 | 1·4491 |
| 0·04 | 0·7078 | 0·9605 | 1·1252 | 1·2385 | 1·3155 | 1·3631 | 1·3884 | 1·3975 | 1·3986 | 1·3986 |
| 0·05 | 0·6826 | 0·9269 | 1·0870 | 1·1976 | 1·2731 | 1·3211 | 1·3471 | 1·3564 | 1·3581 | 1·3582 |
| 0·06 | 0·6613 | 0·8985 | 1·0546 | 1·1632 | 1·2377 | 1·2859 | 1·3123 | 1·3226 | 1·3241 | 1·3244 |
| 0·07 | 0·6428 | 0·8741 | 1·0267 | 1·1332 | 1·2071 | 1·2551 | 1·2823 | 1·2930 | 1·2947 | 1·2947 |
| 0·08 | 0·6264 | 0·8526 | 1·0021 | 1·1070 | 1·1800 | 1·2278 | 1·2554 | 1·2668 | 1·2688 | 1·2688 |
| 0·09 | 0·6118 | 0·8331 | 0·9799 | 1·0833 | 1·1557 | 1·2035 | 1·2313 | 1·2431 | 1·2452 | 1·2452 |
| 0·10 | 0·5985 | 0·8155 | 0·9597 | 1·0618 | 1·1334 | 1·1812 | 1·2092 | 1·2216 | 1·2238 | 1·2238 |
| 0·15 | 0·5450 | 0·7443 | 0·8784 | 0·9746 | 1·0438 | 1·0914 | 1·1208 | 1·1347 | 1·1378 | 1·1279 |
| 0·20 | 0·5045 | 0·6905 | 0·8168 | 0·9086 | 0·9756 | 1·0230 | 1·0533 | 1·0687 | 1·0726 | 1·0728 |
| 0·25 | 0·4714 | 0·6465 | 0·7664 | 0·8544 | 0·9196 | 0·9666 | 0·9976 | 1·0142 | 1·0190 | 1·0192 |
| 0·30 | 0·4431 | 0·6088 | 0·7231 | 0·8078 | 0·8715 | 0·9180 | 0·9496 | 0·9673 | 0·9728 | 0·9731 |
| 0·35 | 0·4181 | 0·5756 | 0·6849 | 0·7666 | 0·8287 | 0·8748 | 0·9068 | 0·9254 | 0·9318 | 0·9321 |
| 0·40 | 0·3957 | 0·5455 | 0·6503 | 0·7293 | 0·7899 | 0·8356 | 0·8678 | 0·8873 | 0·8944 | 0·8948 |
| 0·45 | 0·3750 | 0·5180 | 0·6185 | 0·6949 | 0·7540 | 0·7992 | 0·8316 | 0·8518 | 0·8597 | 0·8602 |
| 0·50 | 0·3559 | 0·4924 | 0·5888 | 0·6627 | 0·7204 | 0·7650 | 0·7975 | 0·8183 | 0·8270 | 0·8276 |
| 0·55 | 0·3379 | 0·4681 | 0·5608 | 0·6322 | 0·6884 | 0·7323 | 0·7649 | 0·7863 | 0·7956 | 0·7964 |
| 0·60 | 0·3207 | 0·4449 | 0·5339 | 0·6029 | 0·6576 | 0·7008 | 0·7334 | 0·7551 | 0·7652 | 0·7662 |
| 0·65 | 0·3041 | 0·4225 | 0·5078 | 0·5743 | 0·6275 | 0·6699 | 0·7023 | 0·7245 | 0·7353 | 0·7365 |
| 0·70 | 0·2878 | 0·4005 | 0·4820 | 0·5460 | 0·5976 | 0·6391 | 0·6713 | 0·6938 | 0·7054 | 0·7067 |
| 0·75 | 0·2715 | 0·3785 | 0·4562 | 0·5176 | 0·5675 | 0·6079 | 0·6398 | 0·6626 | 0·6748 | 0·6764 |
| 0·80 | 0·2550 | 0·3559 | 0·4297 | 0·4883 | 0·5363 | 0·5756 | 0·6069 | 0·6300 | 0·6428 | 0·6448 |
| 0·85 | 0·2376 | 0·3321 | 0·4015 | 0·4571 | 0·5029 | 0·5409 | 0·5716 | 0·5946 | 0·6082 | 0·6106 |
| 0·90 | 0·2182 | 0·3054 | 0·3699 | 0·4219 | 0·4652 | 0·5014 | 0·5311 | 0·5540 | 0·5683 | 0·5712 |
| 0·95 | 0·1937 | 0·2718 | 0·3299 | 0·3771 | 0·4167 | 0·4504 | 0·4786 | 0·5010 | 0·5160 | 0·5196 |