בס״ד

1. לוח הנתון :

3	8	11
6	4	10
9	12	21

$$\binom{n_{..}}{n_{1.}} = \binom{21}{12} = 293{,}930$$

לוחות האפשריים עם אותם שוליים :

$\binom{n_{0.}}{n_{.1}-n_{11}}\binom{n_{1.}}{n_{11}}\left(\frac{1}{2}\right)^{n_{11}}$	$\dfrac{\binom{n_{0.}}{n_{.1}-n_{11}}\binom{n_{1.}}{n_{11}}}{\binom{n_{..}}{n_{1.}}}$	לוח
$(1)(10)(\frac{1}{2})^1 = 5.00$	$\binom{11}{11}\binom{10}{1}/\binom{21}{12}$ $= (1)(10)/293{,}930 = 0.0000$	0 11 9 1
$(11)(45)(\frac{1}{2})^2 = 123.75$	$\binom{11}{10}\binom{10}{2}/\binom{21}{12}$ $= (11)(45)/293{,}930 = 0.0017$	1 10 8 2
$(55)(120)(\frac{1}{2})^3 = 825.00$	$\binom{11}{9}\binom{10}{3}/\binom{21}{12}$ $= (55)(120)/293{,}930 = 0.0225$	2 9 7 3
$(165)(210)(\frac{1}{2})^4 = 2165.63$	$\binom{11}{8}\binom{10}{4}/\binom{21}{12}$ $= (165)(210)/293{,}930 = 0.1179$	3 8 ← 6 4
$(330)(252)(\frac{1}{2})^5 = 2598.75$	$\binom{11}{7}\binom{10}{5}/\binom{21}{12}$ $= (330)(252)/293{,}930 = 0.2829$	4 7 5 5
$(426)(210)(\frac{1}{2})^6 = 1397.81$	$\binom{11}{6}\binom{10}{6}/\binom{21}{12}$ $= (462)(210)/293{,}930 = 0.3301$	5 6 4 6
$(462)(120)(\frac{1}{2})^7 = 433.13$	$\binom{11}{5}\binom{10}{7}/\binom{21}{12}$ $= (462)(120)/293{,}930 = 0.1886$	6 5 3 7
$(330)(45)(\frac{1}{2})^8 = 58.01$	$\binom{11}{4}\binom{10}{8}/\binom{21}{12}$ $= (330)(45)/293{,}930 = 0.0505$	7 4 2 8
$(165)(10)(\frac{1}{2})^9 = 3.22$	$\binom{11}{3}\binom{10}{9}/\binom{21}{12}$ $= (165)(10)/293{,}930 = 0.0056$	8 3 1 9
$(55)(1)(\frac{1}{2})^{10} = 0.05$	$\binom{11}{2}\binom{10}{10}/\binom{21}{12}$ $= (55)(1)/293{,}930 = 0.0002$	9 2 0 10

7,610.35 = סכום

עבור $H_0: \omega \le 1$ מול $H_0: \omega > 1$:

$$p\text{-value} = 0.1179 + 0.2829 + 0.3301 + 0.1886 + 0.0505 + 0.0056 + 0.0002$$
$$= 0.9758$$

עבור $H_0: \omega \ge 1$ מול $H_0: \omega < 1$:

$$p\text{-value} = 0.1179 + 0.0225 + 0.0017 + 0.0000 = 0.1421$$

עבור $H_0: \omega = 1$ מול $H_0: \omega \ne 1$:

$$p\text{-value} = 0.0000 + 0.0017 + 0.0225 + 0.1179$$
$$+ 0.0505 + 0.0056 + 0.0002 = 0.1984$$

עבור $H_0: \omega \le \frac{1}{2}$ מול $H_0: \omega > \frac{1}{2}$:

$$p\text{-value} = \frac{1}{7{,}610.35}\left[2165.63 + 2598.75 + 1397.81 + 433.13 + 58.01 + 3.22 + 0.05\right] = 0.8747$$

עבור $H_0: \omega \ge \frac{1}{2}$ מול $H_0: \omega < \frac{1}{2}$:

$$p\text{-value} = \frac{1}{7{,}610.35}\left[2165.63 + 825.00 + 123.75 + 5.00\right] = 0.4099$$

עבור $H_0: \omega = \frac{1}{2}$ מול $H_0: \omega \ne \frac{1}{2}$:

$$p\text{-value} = \frac{1}{7{,}610.35}\left[5.00 + 123.75 + 825.00 + 2165.63 + 1397.81 + 433.13 + 58.01 + 3.22 + 0.05\right]$$
$$= 0.8163$$

The FREQ Procedure

Statistics for Table of a by b

Statistic	DF	Value	Prob
Chi-Square	1	2.2909	0.1301
Likelihood Ratio Chi-Square	1	2.3309	0.1268
Continuity Adj. Chi-Square	1	1.1494	0.2837
Mantel-Haenszel Chi-Square	1	2.1818	0.1396
Phi Coefficient		-0.3303	
Contingency Coefficient		0.3136	
Cramer's V		-0.3303	

WARNING: 50% of the cells have expected counts less than 5. Chi-Square may not be a valid test.

Fisher's Exact Test

Cell (1,1) Frequency (F)	3
Left-sided Pr <= F	0.1421
Right-sided Pr >= F	0.9758
Table Probability (P)	0.1179
Two-sided Pr <= P	0.1984

Estimates of the Relative Risk (Row1/Row2)

Type of Study	Value	95% Confidence Limits	
Case-Control (Odds Ratio)	0.2500	0.0400	1.5637
Cohort (Col1 Risk)	0.4545	0.1529	1.3515
Cohort (Col2 Risk)	1.8182	0.7842	4.2155

Odds Ratio (Case-Control Study)

Odds Ratio	0.2500
Asymptotic Conf Limits	
95% Lower Conf Limit	0.0400
95% Upper Conf Limit	1.5637
Exact Conf Limits	
95% Lower Conf Limit	0.0269
95% Upper Conf Limit	2.1007

Sample Size = 21

```
% Program to compute MLE of odds ratio in 2x2 table
% Assumes that the user has previously input a vector nv with the cell counts

n00=nv(1,1);
n01=nv(1,2);
n10=nv(1,3);
n11=nv(1,4);

r0 = n00 + n01;
r1 = n10 + n11;
c0 = n00 + n10;
c1 = n01 + n11;

kmin = max(c1-r0,0);
kmax = min(r1,c1);

kv = kmin:kmax;
kv2 = kv.^2;

comb1=[];
comb2=[];
for k = kmin:kmax;
  comb1 = [comb1 nchoosek(r0,c1-k)];
  comb2 = [comb2 nchoosek(r1,k)];
 end;
comb = comb1.*comb2;

omega=1;
psi=log(omega);
psi_vec=psi;
diff=Inf;
eps=0.0001;

while (diff > eps);
  ppvec=exp(psi.*kv);
  ptrms=comb.*ppvec;
  prob=ptrms/sum(ptrms);
  e1=sum(kv.*prob);
  e2=sum(kv2.*prob);
  var=e2-e1^2;
  step=(n11-e1)/var;
  psi=psi+step;
  diff=abs(step);
  psi_vec=vertcat(psi_vec,psi);
 end;

siz=size(psi_vec);
nitr=siz(1,1)-1;
results = [(0:nitr)' psi_vec exp(psi_vec)]
sd_phi_hat = sqrt(1/var)

%*******************************************************************************

>> nv=[3 8 6 4];
>> bin

results =
```

	$\hat{\psi}^{(m)}$	$\hat{\omega}^{(m)}$
0	0	1.0000
1.0000	-1.2727	0.2801
2.0000	-1.3147	0.2686
3.0000	-1.3149	0.2685
4.0000	-1.3149	0.2685

```
sd_phi_hat =
    0.9073
```

בס"ד

3. נתחיל עם הביטוי עבור מקרה רטרוספקטיבי ונפשט את הענין:

$$\Pr(n_{11}=k \mid \text{שוליים}) = \binom{n_{\cdot 0}}{n_{1\cdot}-k}\binom{n_{\cdot 1}}{k}\omega^k \Big/ \sum_{l}\binom{n_{\cdot 0}}{n_{1\cdot}-l}\binom{n_{\cdot 1}}{l}\omega^l$$

$$= \frac{\cancel{n_{\cdot 0}!}}{(n_{1\cdot}-k)!\,(n_{\cdot 0}-n_{1\cdot}+k)!}\;\frac{\cancel{n_{\cdot 1}!}}{k!\,(n_{\cdot 1}-k)!}\;\omega^k$$

$$\Big/ \sum_{l}\frac{\cancel{n_{\cdot 0}!}}{(n_{1\cdot}-l)!\,(n_{\cdot 0}-n_{1\cdot}+l)!}\;\frac{\cancel{n_{\cdot 1}!}}{l!\,(n_{1\cdot}-l)!}\;\omega^l$$

$$= \frac{1}{k!\,(n_{1\cdot}-k)!}\;\frac{1}{(n_{\cdot 1}-k)!\,(n_{\cdot 0}-n_{1\cdot}+k)!}\;\omega^k$$

$$\Big/ \sum_{l}\frac{1}{l!\,(n_{1\cdot}-l)!}\;\frac{1}{(n_{\cdot 1}-l)!\,(n_{\cdot 0}-n_{1\cdot}+l)!}\;\omega^l$$

$$= \underbrace{\frac{n_{1\cdot}!}{k!\,(n_{1\cdot}-k)!}}_{=\binom{n_{1\cdot}}{k}}\;\frac{n_{0\cdot}!}{(n_{\cdot 1}-k)!\,(n_{\cdot 0}-n_{1\cdot}+k)!}\;\omega^k$$

$$\Big/ \sum_{l}\underbrace{\frac{n_{1\cdot}!}{l!\,(n_{1\cdot}-l)!}}_{=\binom{n_{1\cdot}}{l}}\;\frac{n_{0\cdot}!}{(n_{\cdot 1}-l)!\,(n_{\cdot 0}-n_{1\cdot}+l)!}\;\omega^l$$

כעת:

$$n_{\cdot 0}-n_{1\cdot} = n_{00}+n_{10}-n_{10}-n_{11}$$
$$= n_{00}-n_{11} = n_{00}+n_{01}-n_{01}-n_{11}$$
$$= n_{0\cdot}-n_{\cdot 1}$$

לכן

$$(n_{\cdot 0}-n_{1\cdot}+l) = (n_{0\cdot}-n_{\cdot 1}+l)$$
$$= (n_{0\cdot}-(n_{\cdot 1}-l))$$

וגם כן

$$\frac{n_{0\cdot}!}{(n_{\cdot 1}-l)!\,(n_{\cdot 0}-n_{1\cdot}+l)!} = \frac{n_{0\cdot}!}{(n_{\cdot 1}-l)!\,(n_{0\cdot}-(n_{\cdot 1}-l))!} = \binom{n_{0\cdot}}{n_{\cdot 1}-l}$$

לפיכך

$$\Pr(n_{11} \mid \text{שוליים}) = \frac{\binom{n_{0\cdot}}{n_{\cdot 1}-k}\binom{n_{1\cdot}}{k}\omega^k}{\sum_{l}\binom{n_{0\cdot}}{n_{\cdot 1}-l}\binom{n_{1\cdot}}{l}\omega^l}$$

מ.ש.ל.

בס'ד

4.

$$\mathcal{L}(\beta) = \log\left[\binom{n_{0\cdot}}{n_{\cdot 1}-n_{11}}\binom{n_{1\cdot}}{n_{11}}\right] + n_{11}\beta - \log \sum_{\ell} \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} e^{\ell\beta}$$

$$\mathcal{L}'(\beta) = -\frac{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell}\,\ell e^{\ell\beta}}{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} e^{\ell\beta}}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{f}{g}\,\frac{g'}{g}}$$

$$\mathcal{L}''(\beta) = -\left[\frac{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell}\ell^2 e^{\ell\beta}}{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} e^{\ell\beta}} - \left(\frac{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell}\,\ell e^{\ell\beta}}{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} e^{\ell\beta}}\right)^2\right]$$

כעת (כמו שעשינו בכיתה)

$$\frac{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell}\,\ell e^{\beta\ell}}{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} e^{\beta\ell}}$$

$$= \frac{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell}\,\ell e^{\beta\ell}}{\Sigma_{\ell'} \binom{n_{0\cdot}}{n_{\cdot 1}-\ell'}\binom{n_{1\cdot}}{\ell'} e^{\beta\ell'}}$$

$$= \Sigma_\ell\, \ell p_\ell = E[N_{11} \mid N_{\cdot 1} = n_{\cdot 1}]$$

כאשר

$$p_\ell = \frac{\binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} e^{\beta\ell}}{\Sigma_{\ell'} \binom{n_{0\cdot}}{n_{\cdot 1}-\ell'}\binom{n_{1\cdot}}{\ell'} e^{\beta\ell'}}$$

$$= \Pr(N_{11} = \ell \mid N_{\cdot 1} = n_{\cdot 1})$$

באופן דומה

$$\frac{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} \ell^2 e^{\beta\ell}}{\Sigma_\ell \binom{n_{0\cdot}}{n_{\cdot 1}-\ell}\binom{n_{1\cdot}}{\ell} e^{\beta\ell}}$$

$$= \Sigma_\ell \ell^2 p_\ell = E[N_{11}^2 \mid N_{\cdot 1} = n_{\cdot 1}]$$

לכן

$$\mathcal{L}''(\beta) = -\{ E[N_{11}^2 \mid N_{\cdot 1} = n_{\cdot 1}] - (E[N_{11} \mid N_{\cdot 1} = n_{\cdot 1}])^2 \}$$

$$= - Var(N_{11} \mid N_{\cdot 1} = n_{\cdot 1})$$